Quantum Computing Review

Tuesday, September 8, 2020
Rutgers University
Yipeng Huang
Today’s goals

Review quantum computing ground rules

Why are these ground rules the way we explain them??
Course logistics

• Key to success in online classes—reach out: video conf., email, Canvas.

• Questions from last week?

• Due Friday, Sept. 11: 1 question, 1 answer. Iterate as a class.

• Due Friday, Sept. 18: read one of four articles. Share sketches / notes.
Due Friday, Sept. 18: read one of four articles. Share sketches / notes.

• Quantum Computer Systems for Scientific Discovery

• Challenges and Opportunities of Near-Term Quantum Computing Systems

• Quantum computer architecture: towards full-stack quantum accelerators

• Quantum Computing: Progress and Prospects (National Academies Report)
Outline

• Course logistics

• *A game*

• 1 qubit states: what is a qubit, basis states, superposition, Bloch sphere

• 1 qubit gates: Pauli-X, Hadamard, Quirk, unitary matrices

• 2 qubit states: basis states, tensor product

• 2 qubit gates: CNOT, H⊗H, entanglement
A game to show classical rules are insufficient

Premises:

• Physicists observe things obeying these rules.
• First, we try modeling these rules with conventional, “classical” intuition, but we will see it fails.
A game

“Teaching quantum information science to high-school and early undergraduate students” Economou, Rudolph, Barnes, Virginia Tech & Imperial College London

Even if you are familiar with quantum computing ground rules, worth revisiting and asking “why are the rules the way they are?”

FIG. 2. Basic properties of NOT gates.
A game

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FIG. 3. SWAP and CNOT gates.
A game

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FIG. 3. SWAP and CNOT gates.
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Even if you are familiar with quantum computing ground rules, worth revisiting and asking “why are the rules the way they are?”

FIG. 4. Exercise with SWAP and CNOT gates.
A game

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FIG. 4. Exercise with SWAP and CNOT gates.
A game

How far can we push this analogy??

FIG. 5. Introducing the Hadamard gate as a box that produces random outputs upon measurement.
A game

How far can we push this analogy??

FIG. 6. Showing that the random output of a Hadamard gate goes away when two of them are stacked together.
A game

How far can we push this analogy??
A game to show classical rules are insufficient

Premises:

• Physicists observe things obeying these rules.

• First, we try modeling these rules with conventional, “classical” intuition, but we will see it fails.

• The states in our game cannot be thought as merely “probabilistic”
Outline

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• A game
  • 1 qubit states: what is a qubit, basis states, superposition, Bloch sphere
  • 1 qubit gates: Pauli-X, Hadamard, Quirk, unitary matrices
  • 2 qubit states: basis states, tensor product
  • 2 qubit gates: CNOT, H⊗H, entanglement
Maybe first ponder: what is a (classical) bit?
Maybe first ponder: what is a (classical) bit?

• Binary abstraction (high/low voltage)

• Encoded at a specific instant in time

• Statistical property of many particles
1 qubit states: what is a qubit?

• A two-level quantum state (distinct from classical state)

• Carries more information than a classical bit
1 qubit states: what is a qubit?

Physically, can be:

- Quantized voltage and current (IBM, Google superconducting qubits)
- Electron spins (Intel solid state qubits)
- Atom energy states (UMD, IonQ ion trap qubits)
- Polarization of light in different directions
- Etc.
1 qubit states: what is a qubit?

The complete description of quantum states and their evolution is the Schrödinger equation, a continuous time, continuous valued PDE.

Qubits are a discrete time, discrete valued abstraction.

If multiple discrete values are possible (e.g., voltage and current, atom energy states), we pick (bottom) two for our abstraction.
1 qubit states: basis states

\[ |0\rangle \]

\[ |1\rangle \]
1 qubit states: superposition

\[
\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \mid 0 \rangle + \frac{1}{\sqrt{2}} \mid 1 \rangle
\]

\[
\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \mid 0 \rangle - \frac{1}{\sqrt{2}} \mid 1 \rangle
\]
1 qubit states: superposition

\[
\begin{bmatrix}
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}}
\end{bmatrix} = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle
\]

\[
\begin{bmatrix}
\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}}
\end{bmatrix} = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle
\]

|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle
1 qubit states: superposition

\[
\begin{bmatrix}
1/\sqrt{2} \\
1/\sqrt{2}
\end{bmatrix} = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle
\]

\[
\begin{bmatrix}
1/\sqrt{2} \\
-1/\sqrt{2}
\end{bmatrix} = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle
\]

\[
|\psi\rangle = a_0|0\rangle + a_1|1\rangle
\]

\(a_0, a_1\) are amplitudes, are complex-valued

\(a_0\) : how much “zeroness”

\(a_1\) : how much “oneness”

\(|a_0|^2 + |a_1|^2 = 1\)
1 qubit states: Bloch sphere

\[ |\psi\rangle = a_0|0\rangle + a_1|1\rangle \]

\( a_0, a_1 \) are amplitudes, are complex-valued

\( a_0 \): how much “zeroness”

\( a_1 \): how much “oneness”

\[ |a_0|^2 + |a_1|^2 = 1 \]

The state of a single qubit is represented by \( |\psi\rangle = a_0|0\rangle + a_1|1\rangle \). The probability condition \( |a_0|^2 + |a_1|^2 = 1 \) restricts the values that \( a_0 \) and \( a_1 \) can take. We can account for this constraint by setting the magnitude of \( a_0 \) to \( \cos \frac{\theta}{2} \) and the magnitude of \( a_1 \) to \( \sin \frac{\theta}{2} \), since \((\sin \frac{\theta}{2})^2 + (\cos \frac{\theta}{2})^2 = 1\). Accounting for the phase component of a complex number means \( a_0 = e^{i\alpha}\cos \frac{\theta}{2} \) and \( a_1 = e^{i(\alpha+\phi)}\sin \frac{\theta}{2} \). As a result, the state of the qubit can be represented using three independent real numbers \( \alpha, \theta, \) and \( \phi \): \( |\psi\rangle = e^{i\alpha}(\cos \frac{\theta}{2}|0\rangle + e^{i\phi}\sin \frac{\theta}{2}|1\rangle) \). It turns out that the global phase \( \alpha \) has no physical significance whatsoever, and a single-qubit state can be fully described by two real numbers \( 0 \leq \theta < \pi \) and \( 0 \leq \phi < 2\pi \). The description of an arbitrary single-qubit state can be mapped onto a point on the surface of a unit sphere (called a “Bloch sphere”), where the north and south pole correspond to the states \( |0\rangle \) and \( |1\rangle \), respectively. \( \theta \) gives the latitude and \( \phi \) gives the longitude of the positive of the quantum state on the Bloch sphere, as shown in Figure 2.3.1.

**FIGURE 2.3.1** A picture of the Bloch sphere, which represents the set of all possible states for a single qubit. The qubit angles \( \theta \) and \( \phi \) are shown in the figure. Single-qubit gates rotate the qubit state to another point on this sphere. SOURCE: Smite-Meister,


1 qubit states: Bloch sphere

What is the significance?

BOX 2.3
Visualizing the State of a Qubit

The state of a single qubit is represented by \(|\psi\rangle = a_0|0\rangle + a_1|1\rangle\). The probability condition \(|a_0|^2 + |a_1|^2 = 1\) restricts the values that \(a_0\) and \(a_1\) can take. We can account for this constraint by setting the magnitude of \(a_0\) to \(\cos \frac{\theta}{2}\) and the magnitude of \(a_1\) to \(\sin \frac{\theta}{2}\), since \((\sin \frac{\theta}{2})^2 + (\cos \frac{\theta}{2})^2 = 1\).

Accounting for the phase component of a complex number means \(a_0 = e^{i\alpha}\cos \frac{\theta}{2}\) and \(a_1 = e^{i(\alpha + \phi)}sin \frac{\theta}{2}\). As a result, the state of the qubit can be represented using three independent real numbers \(\alpha, \theta, \) and \(\phi\): \(|\psi\rangle = e^{i\alpha}(\cos \frac{\theta}{2}|0\rangle + e^{i\phi}\sin \frac{\theta}{2}|1\rangle\)). It turns out that the global phase \(\alpha\) has no physical significance whatsoever, and a single-qubit state can be fully described by two real numbers \(0 \leq \theta < \pi\) and \(0 \leq \phi < 2\pi\). The description of an arbitrary single-qubit state can be mapped onto a point on the surface of a unit sphere (called a “Bloch sphere”), where the north and south pole correspond to the states \(|0\rangle\) and \(|1\rangle\), respectively. \(\theta\) gives the latitude and \(\phi\) gives the longitude of the positive of the quantum state on the Bloch sphere, as shown in Figure 2.3.1.

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1 qubit states: Bloch sphere

What is the significance?

φ signifies relative phase between “zero” component and “one” component.

At |0⟩ and |1⟩, no meaningful relative phase φ.

|a₀|² + |a₁|² = 1 constraint puts |ψ⟩ = a₀|0⟩ + a₁|1⟩ vector on surface of unit sphere.

The locations of |0⟩ and |1⟩ are not special.

The state of a single qubit is represented by |ψ⟩ = a₀|0⟩ + a₁|1⟩. The probability condition |a₀|² + |a₁|² = 1 restricts the values that a₀ and a₁ can take. We can account for this constraint by setting the magnitude of a₀ to \( \cos \frac{\theta}{2} \) and the magnitude of a₁ to \( \sin \frac{\theta}{2} \), since \((\sin \frac{\theta}{2})² + (\cos \frac{\theta}{2})² = 1\). Accounting for the phase component of a complex number means \(a₀ = e^{ia \cos \frac{\theta}{2}}\) and \(a₁ = e^{i(\alpha + \phi) \sin \frac{\theta}{2}}\). As a result, the state of the qubit can be represented using three independent real numbers \(a, \theta,\) and \(\phi\): \(|ψ⟩ = e^{ia}(\cos \frac{\theta}{2}|0⟩ + e^{i\phi} \sin \frac{\theta}{2}|1⟩)\). It turns out that the global phase \(a\) has no physical significance whatsoever, and a single-qubit state can be fully described by two real numbers \(0 \leq \theta < \pi\) and \(0 \leq \phi < 2\pi\). The description of an arbitrary single-qubit state can be mapped onto a point on the surface of a unit sphere (called a “Bloch sphere”), where the north and south pole correspond to the states |0⟩ and |1⟩, respectively. \(\theta\) gives the latitude and \(\phi\) gives the longitude of the positive of the quantum state on the Bloch sphere, as shown in Figure 2.3.1.

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• A game
• 1 qubit states: what is a qubit, basis states, superposition, Bloch sphere
  • 1 qubit gates: Pauli-X, Hadamard, Quirk, unitary matrices
• 2 qubit states: basis states, tensor product
• 2 qubit gates: CNOT, H⊗H, entanglement
One qubit gates: Pauli-X gate

What would represent X?

FIG. 2. Basic properties of NOT gates.
One qubit gates: Pauli-X gate

\[ X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

\[ X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle \]

\[ X|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle \]

\[ XX|0\rangle = X|1\rangle = |0\rangle \]

\[ XX|1\rangle = X|0\rangle = |1\rangle \]

\[ XX = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \]
One qubit gates: Hadamard gate

What would represent H?
One qubit gates: Hadamard gate

\[ H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \]

\[ H|0\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \]

\[ H|1\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \]

FIG. 7. Hadamard gates produce more balanced input states.
One qubit gates: Hadamard gate

\[ HH|0\rangle = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = [1] = |0\rangle \]

\[ HH|1\rangle = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = [0] = |1\rangle \]

\[ HH = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = [1 \ 0 \ 0 \ 1] = I \]
https://algassert.com/quirk

• By Google researcher Craig Gidney

• Let’s repeat the XX example, HH example

• Let’s try Z, and HZH

• Let’s try S, and HSH
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• 2 qubit gates: CNOT, H⊗H, entanglement
2 qubit states: basis states, tensor product

\[ |0\rangle \otimes |0\rangle = [1] \otimes [1] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = |00\rangle \]

\[ |0\rangle \otimes |1\rangle = [1] \otimes [0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = |01\rangle \]

\[ |1\rangle \otimes |0\rangle = [0] \otimes [1] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = |10\rangle \]

\[ |1\rangle \otimes |1\rangle = [0] \otimes [0] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = |11\rangle \]
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2 qubit gates: CNOT

\[
\text{CNOT} |00\rangle = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = |00\rangle = |0\rangle \otimes |0\rangle
\]

\[
\text{CNOT} |01\rangle = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = |01\rangle = |0\rangle \otimes |1\rangle
\]

\[
\text{CNOT} |10\rangle = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = |11\rangle = |1\rangle \otimes |0\rangle
\]

\[
\text{CNOT} |11\rangle = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = |10\rangle = |1\rangle \otimes |1\rangle
\]
2 qubit gates: $H \otimes H$

\[
H \otimes H |11\rangle = \begin{bmatrix}
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}}
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
1 \\
1
\end{bmatrix} = \begin{bmatrix}
1/2 & 1/2 & 1/2 & 1/2 \\
1/2 & -1/2 & 1/2 & -1/2 \\
1/2 & 1/2 & -1/2 & -1/2 \\
1/2 & -1/2 & -1/2 & 1/2
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
1 \\
1
\end{bmatrix} = \begin{bmatrix}
1/2 \\
-1/2 \\
-1/2 \\
1/2
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}}
\end{bmatrix} = H|1\rangle \otimes H|1\rangle
\]
2 qubit gates: entanglement

\[
\text{CNOT}(\text{H} \otimes I|00\rangle) = \text{CNOT}(\text{H}|0\rangle \otimes I|0\rangle) = \text{CNOT}
\begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{1} \\
\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1 \\
\end{bmatrix} =
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\frac{1}{\sqrt{2}} \\
0 \\
\frac{1}{\sqrt{2}} \\
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{\sqrt{2}} \\
0 \\
\frac{1}{\sqrt{2}} \\
\end{bmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle
\]
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