

Languages and representations for quantum computing: stabilizers and tensor networks

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Where we are in this class

- ▶ Near-term quantum algorithms: QAOA, VQE.
- ▶ Quantum programming languages:
Cirq, Qiskit, assertions, verification.
- ▶ Quantum supremacy: prototyping vs. simulation.
- ▶ Quantum architectural interface:
universal logical gatesets & machine native gatesets.
- ▶ Quantum micro-architecture & devices.

Importance of representations in quantum intuition, programming, and simulation

- ▶ Conventional quantum circuits and state vector view of QC conceals symmetries, hinders intuition.
- ▶ Classical simulation of quantum computing is actually tractable for *a certain subset* of quantum gates.
- ▶ Both the logical and native gatesets in a quantum architecture need to be universal for quantum advantage.

Several views/representations of quantum computing

- ▶ Programming has several views: functional programming, procedural programming.
- ▶ Physics has several views: Newtonian, Lagrangian, Hamiltonian

Different views reveal different symmetries, offer different intuition.

Several views/representations of quantum computing

- ▶ Schrödinger: state vectors and density matrices
- ▶ Heisenberg: stabilizer formalism
- ▶ Tensor-network
- ▶ Feynman: path sums

A survey of these representations of quantum computing is given in Chapter 9 of this recent book [Ding and Chong, 2020].

Several views/representations of quantum computing

- ▶ Schrödinger: state vectors and density matrices
- ▶ Heisenberg: stabilizer formalism
- ▶ Tensor-network
- ▶ Feynman: path sums
- ▶ Binary decision diagrams (new?)
- ▶ Logical satisfiability equations (new? see 10/28/2020 seminar)

Schrödinger view

- ▶ In Schrödinger quantum mechanics description, emphasis on how *states* evolve.

- ▶ $CNOT_{0,1}(H_0 \otimes I_1) |00\rangle =$

$$CNOT_{0,1} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} =$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

- ▶ The Schrodinger view requires exponential storage: A quantum computer with N qubits can be in superposition of 2^N basis states, requires 2^N amplitudes to fully specify state.

Heisenberg view / stabilizer formalism

- ▶ In Heisenberg quantum mechanics description, emphasis on how *operators* evolve.
- ▶ If we limit operations to the Clifford gates (a subset of quantum gates), simulation tractable in polynomial time and space.
- ▶ Covers some quantum algorithms: quantum superdense coding, quantum teleportation, Deutsch-Jozsa, Bernstein-Vazirani, quantum error correction, most quantum error correction protocols.
- ▶ A model for probabilistic (but not quantum) computation.

General strategy for using stabilizers to simulate quantum circuits consisting of only Clifford gates

1. Start with N qubits with initial state $|0\rangle^{\otimes N}$.
2. Represent the state as its group of stabilizers.
3. When simulating the quantum circuit, decompose the Clifford gates to stabilizer gates $\{CNOT, H, P\}$.
4. Apply each of the stabilizer gates to the stabilizer representation.

Special places on the Bloch sphere

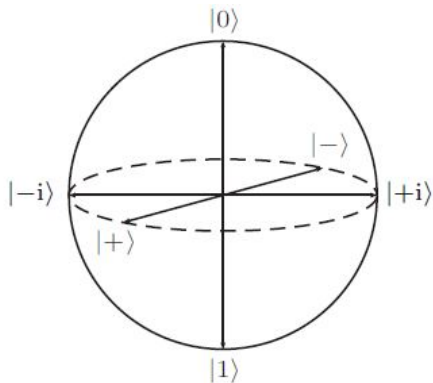


Figure: Bloch sphere showing pole states. Source: Wikimedia.

Representing a state as its group of stabilizers

- ▶ A unitary operator U stabilizes a pure state $|\psi\rangle$ if $U|\psi\rangle = |\psi\rangle$

Representing a state as its group of stabilizers

1. I stabilizes everything.
2. $-I$ stabilizes nothing.

3. X stabilizes $|+\rangle$: $X|+\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |+\rangle$

4. $-X$ stabilizes $|-\rangle$: $-X|-\rangle = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |-\rangle$

5. Y stabilizes $|+i\rangle$: $Y|+i\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = |+i\rangle$

6. $-Y$ stabilizes $|-i\rangle$: $-Y|-i\rangle = -\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{bmatrix} = |-i\rangle$

7. Z stabilizes $|0\rangle$: $Z|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$

8. $-Z$ stabilizes $|1\rangle$: $-Z|1\rangle = -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$

Representing a state as its group of stabilizers

In other words,

1. $|0\rangle$ is stabilized by $\{I, Z\}$
2. $|1\rangle$ is stabilized by $\{I, -Z\}$
3. $|+\rangle$ is stabilized by $\{I, X\}$
4. $|-\rangle$ is stabilized by $\{I, -X\}$
5. $|+i\rangle$ is stabilized by $\{I, Y\}$
6. $|-i\rangle$ is stabilized by $\{I, -Y\}$

Representing a state as its group of stabilizers

1. $|0\rangle$ is stabilized by $\{I, Z\}$
 2. $|1\rangle$ is stabilized by $\{I, -Z\}$
 3. $|+\rangle$ is stabilized by $\{I, X\}$
 4. $|-\rangle$ is stabilized by $\{I, -X\}$
 5. $|+i\rangle$ is stabilized by $\{I, Y\}$
 6. $|-i\rangle$ is stabilized by $\{I, -Y\}$
- ▶ The set of unitary matrices that stabilize $|\psi\rangle$ form a group.
 - ▶ $U^\dagger U |\psi\rangle = U^\dagger |\psi\rangle = |\psi\rangle$
 - ▶ if $V |\psi\rangle = |\psi\rangle$ then $UV |\psi\rangle = |\psi\rangle$ and $VU |\psi\rangle = |\psi\rangle$

Representing a state as its group of stabilizers

1. $|0\rangle$ is stabilized by $\{I, Z\}$
2. $|1\rangle$ is stabilized by $\{I, -Z\}$
3. $|+\rangle$ is stabilized by $\{I, X\}$
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5. $|+i\rangle$ is stabilized by $\{I, Y\}$
6. $|-i\rangle$ is stabilized by $\{I, -Y\}$

Representing a state as its group of stabilizers

For multi-qubit states, the group of stabilizers is the cartesian product of the single-qubit stabilizers

- ▶ $|00\rangle = |0\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, Z \otimes I, Z \otimes Z\}$
- ▶ $\frac{|00\rangle + |10\rangle}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle = |+\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, X \otimes I, X \otimes Z\}$
- ▶ $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ (which cannot be factored into a tensor product) is stabilized by $\{I \otimes I, X \otimes X, -Y \otimes Y, Z \otimes Z\}$

Representing a state as its group of stabilizers

For multi-qubit states, the group of stabilizers is the cartesian product of the single-qubit stabilizers

- ▶ $|0\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, Z \otimes I, Z \otimes Z\}$
- ▶ $|+\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, X \otimes I, X \otimes Z\}$

The same (abelian) group properties hold.

- ▶ The set of unitary matrices that stabilize $|\psi\rangle$ form a group.
- ▶ $U^\dagger U |\psi\rangle = U^\dagger |\psi\rangle = |\psi\rangle$
- ▶ if $V |\psi\rangle = |\psi\rangle$ then $UV |\psi\rangle = |\psi\rangle$ and $VU |\psi\rangle = |\psi\rangle$

Representing a state as its group of stabilizers

Amazing fact: for any N-qubit stabilized state, only N elements needed to specify group.

1. $|0\rangle$ is stabilized by $\{I, Z\}$, Z is generator
2. $|1\rangle$ is stabilized by $\{I, -Z\}$, $-Z$ is generator
3. $|+\rangle$ is stabilized by $\{I, X\}$, X is generator
4. $|-\rangle$ is stabilized by $\{I, -X\}$, $-X$ is generator
5. $|+i\rangle$ is stabilized by $\{I, Y\}$, Y is generator
6. $|-i\rangle$ is stabilized by $\{I, -Y\}$, $-Y$ is generator
7. $|0\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, Z \otimes I, Z \otimes Z\}$,
 $\{I \otimes Z, Z \otimes I\}$ is generator
8. $|+\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, X \otimes I, X \otimes Z\}$,
 $\{I \otimes Z, X \otimes I\}$ is generator
9. $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ is stabilized by $\{I \otimes I, X \otimes X, -Y \otimes Y, X \otimes Z\}$,
 $\{X \otimes X, Z \otimes Z\}$ is generator

Representing a state as its group of stabilizers

- ▶ Amazing fact: for any N -qubit stabilized state, only N elements needed to specify group—a result from abstract algebra group theory [Nielsen and Chuang, 2002, Appendix 2]
- ▶ So long as the quantum circuit consists only of Clifford gates, only N elements needed to specify whole quantum state.
- ▶ Contrast against 2^N amplitudes needed to specify a general N -qubit quantum state vector.
- ▶ For example a two-qubit states needs four amplitues $\{a_0, a_1, a_2, a_3\}$ to specify quantum state $|\psi\rangle = a_0 |00\rangle + a_1 |01\rangle + a_2 |10\rangle + a_3 |11\rangle$.

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4. Apply each of the stabilizer gates to the stabilizer representation.

Stabilizer gates: $\{CNOT, H, P\}$

1. Hadamard gate: induces superpositions.
 2. CNOT gate: induces entanglement.
 3. Phase gate: induces complex phases. $P = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
- ▶ Despite featuring superposition, entanglement, and complex amplitudes, is *not* universal for quantum computing.
 - ▶ We shall see that the deeply symmetrical structure of these gates prevent access to full quantum Hilbert space.

Stabilizer gates are a generator for Pauli gates (i.e., Clifford gates decompose to stabilizer gates)

Pauli gates are rotations around respective axes by π .

$$\blacktriangleright Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = PP$$

$$\blacktriangleright X = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = HZH$$

$$\blacktriangleright Y = iXZ$$

$$\blacktriangleright X^2 = Y^2 = Z^2 = I$$

\blacktriangleright Symmetry is similar to quaternions.

\blacktriangleright With Clifford gates consisting of $\{CNOT, H, P, I, X, Y, Z\}$, sufficient to build many quantum algorithms, including: quantum superdense coding, quantum teleportation, Deutsch-Jozsa, Bernstein-Vazirani, quantum error correction, most quantum error correction protocols.

General strategy for using stabilizers to simulate quantum circuits consisting of only Clifford gates

1. Start with N qubits with initial state $|0\rangle^{\otimes N}$.
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Single qubit stabilizer gates bounce stabilizer states around an octahedron on the Bloch sphere

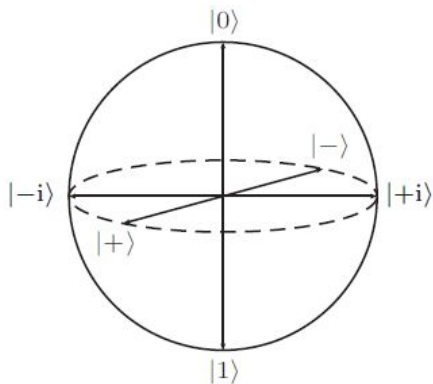


Figure: Bloch sphere showing pole states. Source: Wikimedia.

Single qubit stabilizer gates bounce stabilizer states around an octahedron on the Bloch sphere

$$H|0\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |+\rangle$$

$$H|1\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = |-\rangle$$

$$H|+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

$$H|-\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

$$H|+i\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix} = |-i\rangle$$

$$H|-i\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = |+i\rangle$$

Single qubit stabilizer gates bounce stabilizer states around an octahedron on the Bloch sphere

$$P|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

$$P|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

$$P|+\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = |+i\rangle$$

$$P|-\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{bmatrix} = |-i\rangle$$

$$P|+i\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = |-\rangle$$

$$P|-i\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |+\rangle$$

Apply each of the stabilizer gates to the stabilizer representation.

► Hadamard:

1. $Z \rightarrow X$
2. $-Z \rightarrow -X$
3. $X \rightarrow Z$
4. $-X \rightarrow -Z$
5. $Y \rightarrow -Y$
6. $-Y \rightarrow Y$

► Phase:

1. $Z \rightarrow Z$
2. $-Z \rightarrow -X$
3. $X \rightarrow Y$
4. $-X \rightarrow -Y$
5. $Y \rightarrow -X$
6. $-Y \rightarrow X$

Apply each of the stabilizer gates to the stabilizer representation.

► CNOT:

1. $X \otimes I \rightarrow X \otimes X$

2. $I \otimes X \rightarrow I \otimes X$

3. $Z \otimes I \rightarrow Z \otimes I$

4. $I \otimes Z \rightarrow Z \otimes Z$

Concrete example on Bell state circuit

$$CNOT_{0,1}(H_0 \otimes I_1) |00\rangle$$

1. Start with N qubits with initial state $|0\rangle^{\otimes N}$.
2. Represent the state as its group of stabilizers— $|00\rangle : \{IZ, ZI\}$
3. When simulating the quantum circuit, decompose the Clifford gates to stabilizer gates $\{CNOT, H, P\}$.
4. Apply each of the stabilizer gates to the stabilizer representation.
 - ▶ Hadamard on first qubit— $|+\rangle |0\rangle : \{IZ, XI\}$
 - ▶ CNOT on both qubits— $\frac{|00\rangle + |11\rangle}{\sqrt{2}} : \{ZZ, XX\}$

Gottesman-Knill theorem and its implications

- ▶ Gottesman-Knill theorem states that there exists a classical algorithm that simulates any stabilizer circuit in polynomial time.
- ▶ Any quantum state created by a Clifford circuit, even if it has lots of superpositions and entanglement, is easy to classically simulate.
- ▶ Quantum computers need at least one non-Clifford gate to achieve universal quantum computation.
- ▶ The T gate, where $TT = P$, $PP = Z$ is one common choice.
- ▶ There are results showing that a quantum circuit is only exponentially hard to simulate w.r.t. the number of T-gates.

References

- ▶ Main sources: [Gottesman, 1998] [Aaronson,]
- ▶ Further reference on separation of probabilistic and quantum computing: [Van Den Nes, 2010]
- ▶ Further reference on applications in classical simulation of Clifford quantum circuits: [Aaronson and Gottesman, 2004]
- ▶ Further reference on applications in classical simulation of general quantum circuits: [Bravyi and Gosset, 2016]

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