Emerging languages and representations for quantum computing: Tensor networks

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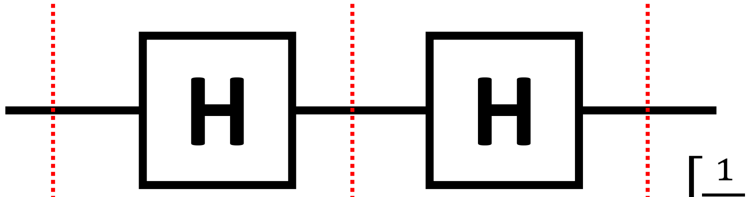
Significance of stabilizers and tensor networks

- Highlight connections between quantum computing semantics to other areas of computer science.
  - Turns quantum computing from an analytical, numerical field into one that has ties to algebra and topology
- Introduce these two important alternative representations for quantum computing
  - Heisenberg view (stabilizer formalism) is important in quantum error correction literature
  - Feynman view (path sums and tensor network contraction) is important in quantum circuit simulation literature

Schrödinger view of quantum computing



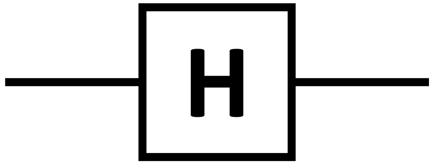
$$\begin{bmatrix}0\\1\end{bmatrix}$$

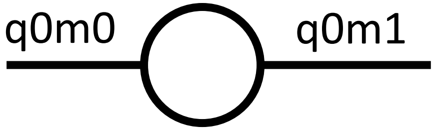
$$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}$$

$$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$

Emphasis is on finding state vectors at each moment.

Feynman (path sum) view of quantum computing





A tensor network

	-
q0m0 =$\vert 0\rangle$	$\begin{bmatrix}0\\1\end{bmatrix}$
q0m0 =$\vert 1\rangle$	

	q0m0 =$\vert 0\rangle$	**q0m0 =$\vert 1\rangle$**
q0m1 =$\vert 0\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
q0m1 =$\vert 1\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$

q0m0	**q0m1**	
$\vert 0\rangle$	$\vert 0\rangle$	$0 \bullet \frac{1}{\sqrt{2}} = 0$
$\vert 0\rangle$	$\vert 1\rangle$	$0 \bullet \frac{1}{\sqrt{2}} = 0$
$\vert 1\rangle$	$\vert 0\rangle$	$1 \bullet \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$
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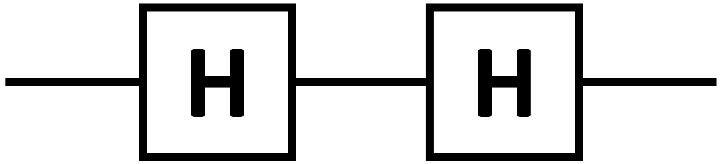
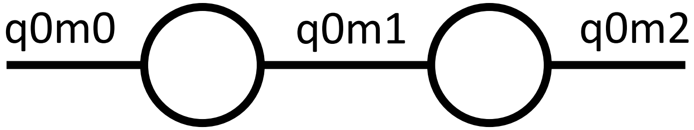
Feynman (path sum) view of quantum computing





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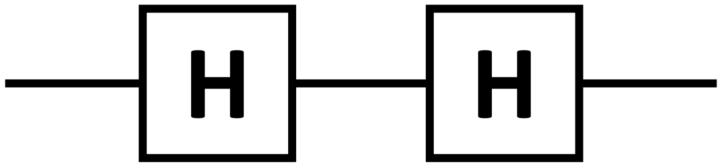
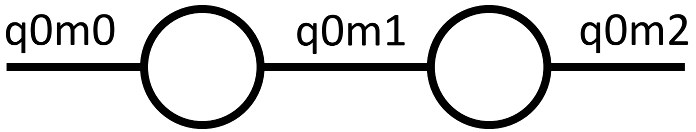
	q0m0 =$\vert 0\rangle$	**q0m0 =$\vert 1\rangle$**
q0m1 =$\vert 0\rangle$	$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}$	
q0m1 =$\vert 1\rangle$		

	q0m1 =$\vert 0\rangle$	**q0m1 =$\vert 1\rangle$**
q0m2 =$\vert 0\rangle$	$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}$	
q0m2 =$\vert 1\rangle$		

Feynman (path sum) view of quantum computing





A tensor network

1. In this calculation technique, we don't form the whole state vector.

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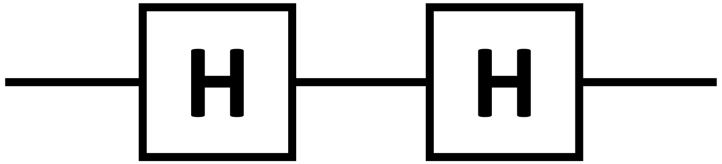
q0m0 $=\|1\rangle$	

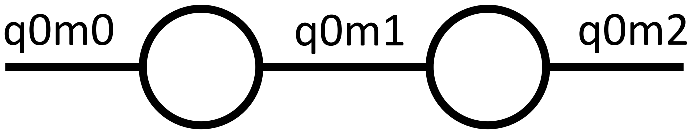
	q0m0 $=\|0\rangle$	q0m0 $=\|1\rangle$
q0m1 $=\|0\rangle$	$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}$	
q0m1 $=\|1\rangle$		

	q0m1 $=\|0\rangle$	q0m1 $=\|1\rangle$
q0m2 $=\|0\rangle$	$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}$	
q0m2 $=\|1\rangle$		

q0m0	q0m1	q0m2	
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = 0$
$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = 0$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = 0$
$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{-1}{\sqrt{2}} = 0$
$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$1 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = \frac{1}{2}$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$1 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = \frac{1}{2}$
$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$1 \bullet \frac{-1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = \frac{-1}{2}$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$1 \bullet \frac{-1}{\sqrt{2}} \bullet \frac{-1}{\sqrt{2}} = \frac{1}{2}$

Feynman (path sum) view of quantum computing





A tensor network

1. In this calculation technique, we don't form the whole state vector.
2. Path sum for q0m2=|0> has destructive interference.

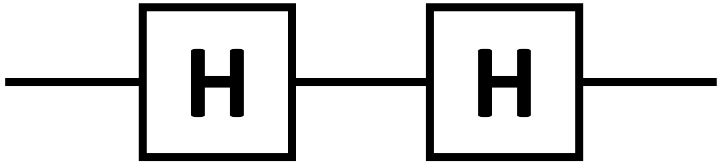
	-
q0m0 $=\|0\rangle$	$\begin{bmatrix}0\\1\end{bmatrix}$
q0m0 $=\|1\rangle$	

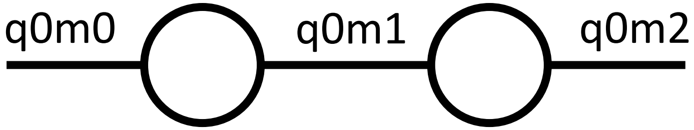
	q0m0 $=\|0\rangle$	q0m0 $=\|1\rangle$
q0m1 $=\|0\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
q0m1 $=\|1\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$

	q0m1 $=\|0\rangle$	q0m1 $=\|1\rangle$
q0m2 $=\|0\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
q0m2 $=\|1\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$

q0m0	q0m1	q0m2	
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = 0$
$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = 0$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = 0$
$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{-1}{\sqrt{2}} = 0$
$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$1 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = \frac{1}{2}$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$1 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = \frac{1}{2}$
$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$1 \bullet \frac{-1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = \frac{-1}{2}$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$1 \bullet \frac{-1}{\sqrt{2}} \bullet \frac{-1}{\sqrt{2}} = \frac{1}{2}$

Feynman (path sum) view of quantum computing





A tensor network

1. In this calculation technique, we don't form the whole state vector.
2. Path sum for q0m2=|0> has destructive interference.
3. Path sum for q0m2=|1> has constructive interference.

q0m0	q0m1	q0m2	
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = 0$
$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = 0$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = 0$
$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{-1}{\sqrt{2}} = 0$
$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$1 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = \frac{1}{2}$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$1 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = \frac{1}{2}$
$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$1 \bullet \frac{-1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = \frac{-1}{2}$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$1 \bullet \frac{-1}{\sqrt{2}} \bullet \frac{-1}{\sqrt{2}} = \frac{1}{2}$

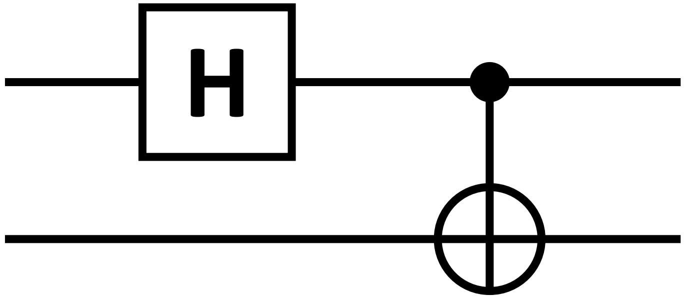
	-
q0m0 $=\|0\rangle$	$\begin{bmatrix}0\\1\end{bmatrix}$
q0m0 $=\|1\rangle$	

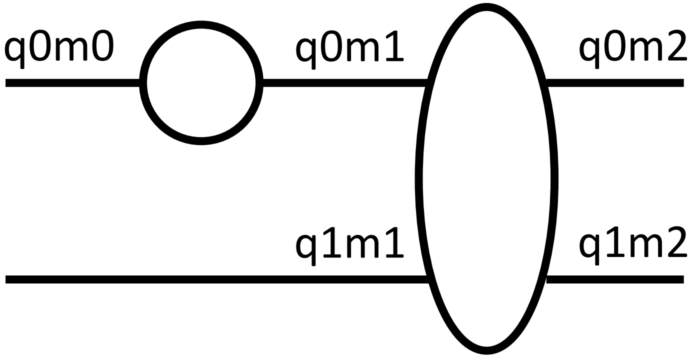
	q0m0 $=\|0\rangle$	q0m0 $=\|1\rangle$
q0m1 $=\|0\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
q0m1 $=\|1\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$

	q0m1 $=\|0\rangle$	q0m1 $=\|1\rangle$
q0m2 $=\|0\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
q0m2 $=\|1\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$

Quantum circuits are a specialized form of tensor network.

- Tensor: a data structure with rank k and dimension m.
  - Rank-0 tensor: scalar.
  - Rank-1 tensor: vector.
  - Rank-2 tensor: matrix.
  - Rank-3 tensor…
- For qubits, dimension m = 2.
  - Rank-1 tensor: a single qubit state.
  - Rank-2 tensor: a 2x2 matrix for a single-qubit gate.
  - Rank-4 tensor: a 2x2x2x2 data structure for a two-qubit gate.





A rank-1 tensor

	-
q0m0 =$\|0\rangle$	$\begin{bmatrix}0\\1\end{bmatrix}$
q0m0 =$\|1\rangle$	

A rank-2 tensor

	q0m0 =$\|0\rangle$	**q0m0 =$\|1\rangle$**
q0m1 =$\|0\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
q0m1 =$\|1\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$

A rank-4 tensor

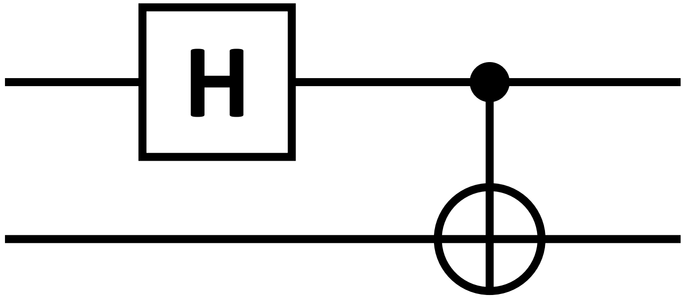
		q0m1=$\|0\rangle$		**q0m1=$\|1\rangle$**	
		q1m1=$\|0\rangle$	**q1m1=$\|1\rangle$**	**q1m1=$\|0\rangle$**	**q1m1=$\|1\rangle$**
q0m2=$\|0\rangle$	**q1m2=$\|0\rangle$**	1	0	0	0
	q1m2=$\|1\rangle$	0	1	0	0
q0m2=$\|1\rangle$	**q1m2=$\|0\rangle$**	0	0	0	1
	q1m2=$\|1\rangle$	0	0	1	0

Tensor network contraction

- Tensor network contraction is one type of tensor-tensor multiplication. It is a generalized form of matrix multiplication.
- Merge two tensors into one. Absorb common edges. If the two tensors share a common index, sum over all possible values of that index.
- For example: contract tensor A and tensor B into tensor C.
    - Where A has rank (x+y), B has rank (y+z), C has rank (x+z)

$$C_{i_1,i_2\ldots,i_x,k_1,k_2\ldots,k_z} = \sum_{j_1,j_2\ldots,j_y \in \{0,..,m-1\}} A_{i_1,i_2\ldots,i_x,j_1,\ldots,j_y} B_{j_1,j_2\ldots,j_y,k_1,\ldots,k_z}$$

Credit: Fried et al. qTorch: The quantum tensor contraction handler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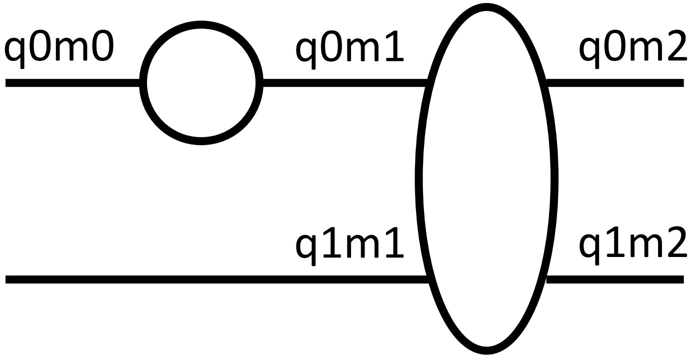
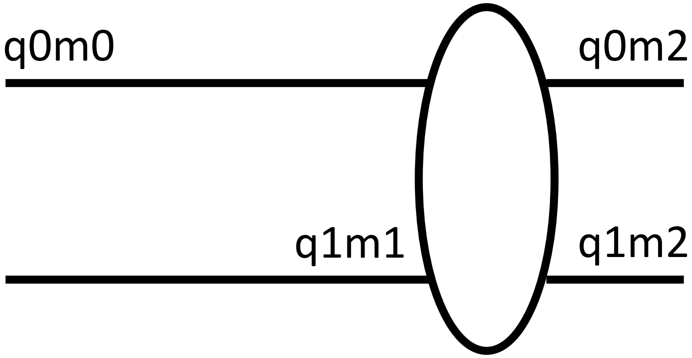

q0m0
q0m1
q0m2
q1m1
q1m2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q1m1
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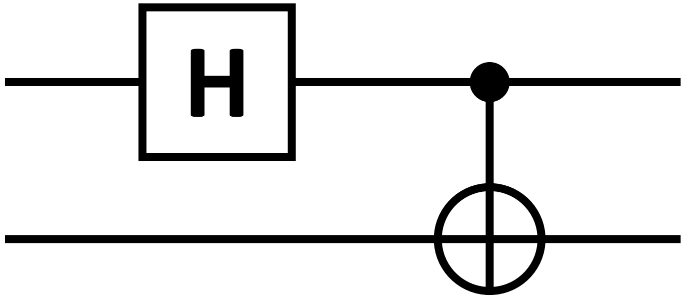
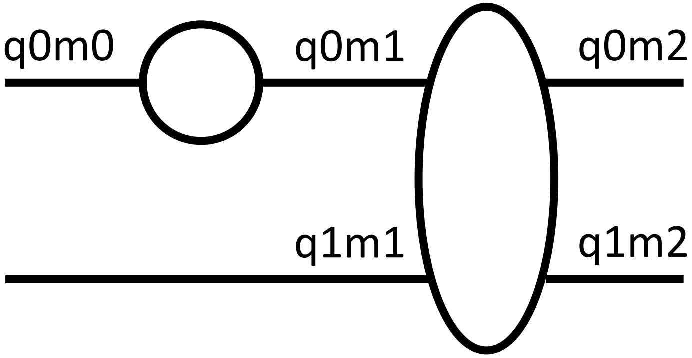
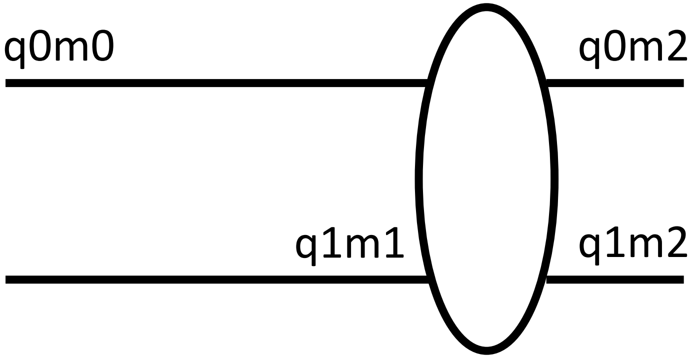


	q0m0 $=\|0\rangle$	q0m0 $=\|1\rangle$
q0m1 $=\|0\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
q0m1 $=\|1\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$



		q0m1 $=\|0\rangle$		q0m1 $=\|1\rangle$	
		q1m1 $=\|0\rangle$	q1m1 $=\|1\rangle$	q1m1 $=\|0\rangle$	q1m1 $=\|1\rangle$
q0m2 $=\|0\rangle$	q1m2 $=\|0\rangle$	1	0	0	0
	q1m2 $=\|1\rangle$	0	1	0	0
q0m2 $=\|1\rangle$	q1m2 $=\|0\rangle$	0	0	0	1
	q1m2 $=\|1\rangle$	0	0	1	0



		q0m0 $=\|0\rangle$		q0m0 $=\|1\rangle$	
		q1m1 $=\|0\rangle$	q1m1 $=\|1\rangle$	q1m1 $=\|0\rangle$	q1m1 $=\|1\rangle$
q0m2 $=\|0\rangle$	q1m2 $=\|0\rangle$	$1/\sqrt{2}$	0	$1/\sqrt{2}$	0
	q1m2 $=\|1\rangle$	0	$1/\sqrt{2}$	0	$1/\sqrt{2}$
q0m2 $=\|1\rangle$	q1m2 $=\|0\rangle$	0	$1/\sqrt{2}$	0	$-1/\sqrt{2}$
	q1m2 $=\|1\rangle$	$1/\sqrt{2}$	0	$-1/\sqrt{2}$	0

Tensor network contraction order

- Contraction ordering says the order in which edges are contracted.
- To minimize computation and memory requirements, best to avoid forming large intermediate tensors.

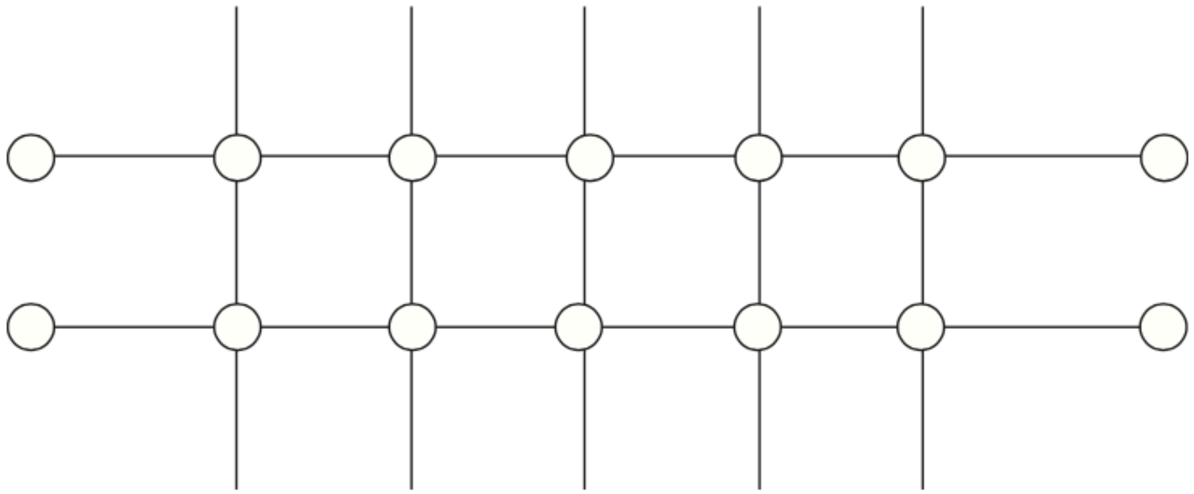


Figure 9.4: Part of a generic tensor network, consisting of ten rank-4 tensors and four rank-1 tensors.

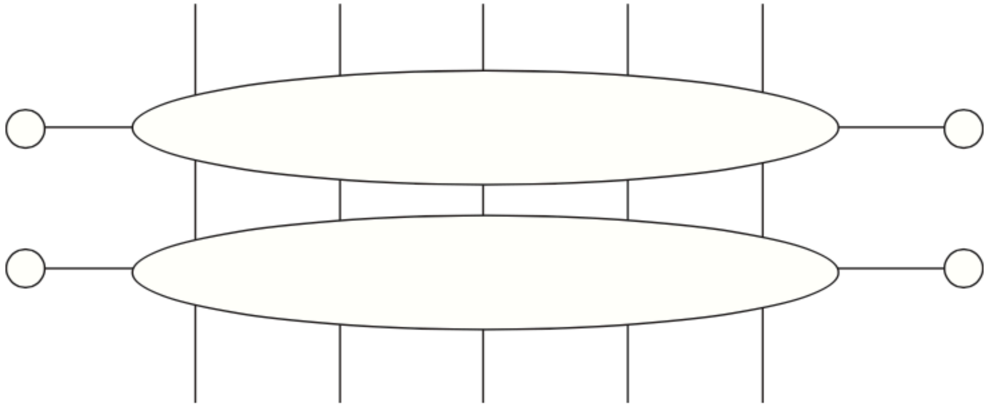


Figure 9.5: First strategy of contraction that results in two rank-12 tensors and four rank-1 tensors. Then contracting the two rank-12 tensors involves contracting 5 edges at once, by summing over 2^5 terms.

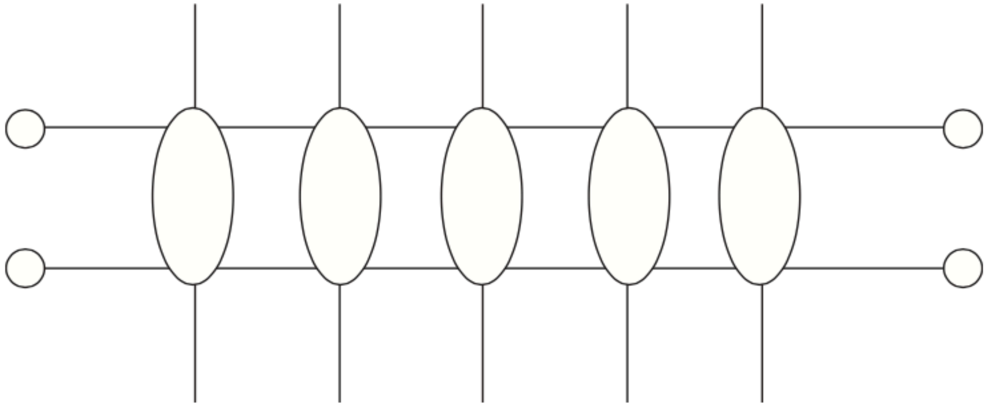


Figure 9.6: Second strategy of contraction that results in five rank-6 tensors and four rank-1 tensors. Then contracting the five rank-6 tensors involves contracting from left to right 2 edges at a time, by summing over 2^2 terms four times.

Ding and Chong. Quantum Computer Systems: Research for Noisy Intermediate-Scale Quantum Computers.

Complexity of quantum circuit simulation via tensor network contraction

- In more detail, cost of simulating the quantum circuit is O(exp(treewidth))
- …where else did we recently see a bound on simulation cost?

Primary Sources

- Markov and Shi. Simulating Quantum Computation by Contracting Tensor Networks. SIAM. 2008.
- Fried, Sawaya, Cao, Kivlichan, Romero, Aspuru-Guzik. qTorch: The quantum tensor contraction handler.
- Biamonte and Bergholm. Quantum Tensor Networks in a Nutshell. 2017.
- Biamonte. Lectures on Quantum Tensor Networks: a pathway to modern diagrammatic reasoning. 2019.