Quantum computing fundamentals: multiple qubits, Deutsch-Jozsa

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Announcements

The class so far

1. Introductions on Canvas discussions. Important for me and classmates to know your interests.
2. Reading: Preskill. "Quantum Computing in the NISQ era and beyond." Describes current state of quantum computing impact and development. Discuss by posting one question and one answer—can be anything.

Intermediate-term class plan

Where we are headed in first month

1. Fundamental rules of quantum computing
2. Basic quantum algorithms
3. Programming examples in Google Cirq
4. A NISQ algorithm: quantum approximate optimization algorithm
5. Programming assignment on QAOA in Cirq

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Universal classical computation

Toffoli (CCNOT) gate can represent all classical computation

(How?)

Universal classical computation

Toffoli (CCNOT) gate can represent all classical computation

1. All Boolean expressions can be phrased as either CNF (and of ors) or DNF (or of ands).
2. AND, OR, and NOT operations are universal.
3. Either NAND or NOR are individually universal.
4. CCNOT implements NAND. (Feed $|1\rangle$ into target qubit). To see this:
   - Write down truth table for NAND.
   - Write down unitary matrix for CCNOT.
   - Write down truth table for CCNOT.
5. So, CCNOT is universal for classical logic.

Superposition

Single qubit state

- $\alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$
- Amplitudes $\alpha, \beta \in \mathbb{C}$
- $|\alpha|^2 + |\beta|^2 = 1$
- The above constraints require that qubit operators are unitary matrices.

Many physical phenomena can be in superposition and encode qubits

- Polarization of light in different directions
- Electron spins (Intel solid state qubits)
- Atom energy states (UMD, IonQ ion trap qubits)
- Quantized voltage and current (IBM, Google superconducting qubits)

If multiple discrete values are possible (e.g., atom energy states, voltage and current), we pick (bottom) two for the binary abstraction.

Superposition

Special names for two common superposition states (with respect to standard basis)

- $|+\rangle = H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
- $|-\rangle = H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Interference

Amplitudes can positively and negatively interfere

- $HH|0\rangle = H\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}+\frac{1}{2}\\ \frac{1}{2}-\frac{1}{2}\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = |0\rangle$
- $HH|1\rangle = H\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}-\frac{1}{2}\\ \frac{1}{2}+\frac{1}{2}\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} = |1\rangle$

Circuit diagram representation:

Measurement

These rules, states, and operators model real quantum phenomena

- States in our examples cannot be merely classical or probabilistic

Let's go through the Q is for Quantum diagrams and complete the circuit diagrams with measurement

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Multiple qubits: the tensor product

Tensor product of unitary matrices

$$X \otimes I \left(\frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{2}} |11\rangle \right) = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} =$$

$$\begin{bmatrix} 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{2}} |11\rangle$$

Circuit diagram representation:

Multiple qubits: the tensor product

Tensor product of unitary matrices

$$H \otimes X\left(\tfrac{1}{2}|00\rangle + \tfrac{1}{2}|01\rangle + \tfrac{1}{2}|10\rangle - \tfrac{1}{2}|11\rangle\right) = \left(\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \frac{-1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} =$$

$$\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

Circuit diagram representation:

Multiple qubits: the tensor product

Tensor product of state vectors

$$X\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes I|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Circuit diagram representation:

Multiple qubits: the tensor product

Tensor product of state vectors

$$|+\rangle \otimes |-\rangle = H|0\rangle \otimes H|1\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \\ \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2} \\ \frac{1}{2} \\ \frac{-1}{2} \end{bmatrix} =$$

$$\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

Circuit diagram representation:

Multiple qubits: the tensor product

Exercise: proof by induction about the Hadamard transform

Show that $|+\rangle^{\otimes n} = \frac{1}{2^{n/2}} \sum_{m=0}^{2^n-1} |m\rangle$

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Deutsch-Jozsa algorithm: simplest quantum algorithm showing advantage vs. classical

A Heist

- You break into a bank vault. The bank vault has 2^N bars. Three possibilities: all are gold, half are gold and half are fake, or all are fake.
- Even if you steal just one gold bar, it is enough to fund your escape from the country, forever evading law enforcement.
- You do not want to risk stealing from a bank vault with only fake bars.
- You have access to an oracle $f(x)$ that tells you if gold bar x is real.
- Using the oracle sounds the alarm, so you only get to use it once.