Quantum computing fundamentals: Deutsch-Jozsa programs and systems

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Announcements

The class so far

1. Be sure to keep up with class via session slides and recommended reading.
2. Practice quantum computing formalism: state vectors, unitary matrices, tensor products.

Intermediate-term class plan

Where we are headed in first month

1. Fundamental rules of quantum computing
2. Basic quantum algorithms
3. Programming examples in Google Cirq
4. A NISQ algorithm: quantum approximate optimization algorithm
5. Programming assignment on QAOA in Cirq

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Superposition

Special names for two common superposition states (with respect to standard basis)

- $|+\rangle = H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
- $|-\rangle = H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Multiple qubits: the tensor product

Exercise: proof by induction about the Hadamard transform

Show that $|+\rangle^{\otimes n} = \frac{1}{2^{n/2}} \sum_{m=0}^{2^n-1} |m\rangle$

Multiple qubits: the tensor product

Exercise: proof by induction about the Hadamard transform

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- Base case $n = 1$: $|+\rangle^{\otimes 1} = |+\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle = \frac{1}{2^{1/2}} \sum_{m=0}^{2^1-1} |m\rangle$

Multiple qubits: the tensor product

Exercise: proof by induction about the Hadamard transform

Show that $|+\rangle^{\otimes n} = \frac{1}{2^{n/2}} \sum_{m=0}^{2^n-1} |m\rangle$

- Base case $n = 1$: $|+\rangle^{\otimes 1} = |+\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle = \frac{1}{2^{1/2}} \sum_{m=0}^{2^1-1} |m\rangle$
- Inductive step assumes statement is true for $n = k - 1$, Then for $n = k$:
  $|+\rangle^{\otimes k} = |+\rangle \otimes |+\rangle^{k-1} = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right) \otimes \frac{1}{2^{(k-1)/2}} \sum_{m=0}^{2^{k-1}-1} |m\rangle =$

$$\frac{1}{2^{1/2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \frac{1}{2^{(k-1)/2}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{2^{k-1} \times 1} = \frac{1}{2^{k/2}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{2^k \times 1}$$

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Deutsch-Jozsa algorithm: simplest quantum algorithm showing advantage vs. classical

A Heist

- You break into a bank vault. The bank vault has 2^n bars. Three possibilities: all are gold, half are gold and half are fake, or all are fake.
- Even if you steal just one gold bar, it is enough to fund your escape from the country, forever evading law enforcement.
- You do not want to risk stealing from a bank vault with only fake bars.
- You have access to an oracle $f(x)$ that tells you if gold bar x is real.
- Using the oracle sounds the alarm, so you only get to use it once.

Deutsch-Jozsa algorithm: simplest quantum algorithm showing advantage vs. classical

More formal description

- The 2^n bars are either fake or gold. $f : \{0,1\}^n \to \{0,1\}$.
- Three possibilities:
  1. All are fake. f is constant. $f(x) = 0$ for all $x \in \{0,1\}^n$.
  2. All are gold. f is constant. $f(x) = 1$ for all $x \in \{0,1\}^n$.
  3. Half fake half gold. f is balanced.
     $$\left|\{x \in \{0,1\}^n : f(x) = 0\}\right| = \left|\{x \in \{0,1\}^n : f(x) = 1\}\right| = 2^{n-1}$$
- The oracle U works as follows: $U\left|c\right\rangle\left|t\right\rangle = \left|c\right\rangle\left|t \oplus f(c)\right\rangle$
- Try deciding if f is constant or balanced using oracle U only once.

Circuit diagram

Compare to diagram in Rudolph, "Q is for Quantum".

What is in the oracle

For $n = 1$, four possibilities

	f_0	f_1	f_2	f_3
f(0)	0	0	1	1
f(1)	0	1	0	1
	f is constant 0	f is balanced	f is balanced	f is constant 1

Demonstration of Deutsch-Jozsa for the $n = 1$ case

Output of circuit is $c = 0$ iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle \otimes |1\rangle = |0\rangle |1\rangle = |01\rangle$
2. After first set of Hadamards: $H \otimes H\Big(|0\rangle \otimes |1\rangle\Big) = H|0\rangle \otimes H|1\rangle = |+\rangle \otimes |-\rangle =$

$$\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

From here, let's take an aside via matrix-vector multiplication to build intuition with interference and phase kickback.

Demonstration of Deutsch-Jozsa for the $n = 1$ case

Output of circuit is $c = 0$ iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle \otimes |1\rangle = |0\rangle\,|1\rangle = |01\rangle$
2. After first set of Hadamards: $H \otimes H\Big(|0\rangle \otimes |1\rangle \Big) = |+\rangle\,|-\rangle =$
$$\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) = \frac{1}{2}\Big(|0\rangle\,(\,|0\rangle - |1\rangle\,) + |1\rangle\,(\,|0\rangle - |1\rangle\,)\Big)$$
3. After applying oracle U:
$$U\frac{1}{2}\Big(|0\rangle\,(\,|0\rangle - |1\rangle\,) + |1\rangle\,(\,|0\rangle - |1\rangle\,)\Big) = \frac{1}{2}\Big(|0\rangle\,(\,|f(0) \oplus 0\rangle - |f(0) \oplus 1\rangle\,) +$$
$$|1\rangle\,(\,|f(1) \oplus 0\rangle - |f(1) \oplus 1\rangle\,)\Big) = \frac{1}{2}\Big(|0\rangle\,(\,|f(0)\rangle - |f(\bar{0})\rangle\,) + |1\rangle\,(\,|f(1)\rangle - |f(\bar{1})\rangle\,)\Big)$$

Demonstration of Deutsch-Jozsa for the $n = 1$ case

Output of circuit is $c = 0$ iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle \otimes |1\rangle = |0\rangle |1\rangle = |01\rangle$
2. After first set of Hadamards: $\frac{1}{2}\Big(|0\rangle (|0\rangle - |1\rangle) + |1\rangle (|0\rangle - |1\rangle) \Big)$
3. After applying oracle U: $U\frac{1}{2}\Big(|0\rangle (|0\rangle - |1\rangle) + |1\rangle (|0\rangle - |1\rangle) \Big) =$
   $\frac{1}{2}\Big(|0\rangle \left(|f(0)\rangle - |\overline{f(0)}\rangle \right) + |1\rangle \left(|f(1)\rangle - |\overline{f(1)}\rangle \right) \Big)$
4. This last expression can be factored depending on f:
   $U\frac{1}{2}\Big(|0\rangle (|0\rangle - |1\rangle) + |1\rangle (|0\rangle - |1\rangle) \Big) =$
   $\begin{cases} \frac{1}{2}\left(|0\rangle + |1\rangle \right)\left(|f(0)\rangle - |\overline{f(0)}\rangle \right) \text{ if } f(0) = f(1) \\ \frac{1}{2}\left(|0\rangle - |1\rangle \right)\left(|f(0)\rangle - |\overline{f(0)}\rangle \right) \text{ if } f(0) \neq f(1) \end{cases} = \begin{cases} |+\rangle |-\rangle \text{ if } f(0) = f(1) \\ |-\rangle |-\rangle \text{ if } f(0) \neq f(1) \end{cases}$

The trick where oracle's output on $|t\rangle$ affects phase of $|c\rangle$ is called phase kickback.

Demonstration of Deutsch-Jozsa for the $n = 1$ case

Output of circuit is $c = 0$ iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle \otimes |1\rangle = |0\rangle |1\rangle = |01\rangle$
2. After first set of Hadamards: $\frac{1}{2}\Big(|0\rangle (|0\rangle - |1\rangle) + |1\rangle (|0\rangle - |1\rangle)\Big)$
3. After applying oracle U:
$$U\frac{1}{2}\Big(|0\rangle (|0\rangle - |1\rangle) + |1\rangle (|0\rangle - |1\rangle)\Big) = \begin{cases} |+\rangle |-\rangle & \text{if } f(0) = f(1) \\ |-\rangle |-\rangle & \text{if } f(0) \neq f(1) \end{cases}$$
4. After applying second H on top qubit:
$$\begin{cases} H \otimes I\big(|+\rangle |-\rangle \big) = |0\rangle |-\rangle & \text{if } f(0) = f(1) \\ H \otimes I\big(|-\rangle |-\rangle \big) = |1\rangle |-\rangle & \text{if } f(0) \neq f(1) \end{cases}$$

Deutsch-Jozsa programs and systems

Algorithm

David Deutsch and Richard Jozsa. Rapid solution of problems by quantum computation. 1992.

Programs

Google Cirq programming example.

Implementation

- Mach-Zehnder interferometer implementation. `https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/SinglePhotonLab/SinglePhotonLab.html`
- Ion trap implementation. Gulde et al. Implementation of the Deutsch–Jozsa algorithm on an ion-trap quantum computer. Letters to Nature. 2003.