Quantum computing fundamentals: Deutsch-Jozsa programs and systems

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Table of contents

Announcements

New reading assignment, choose one

1. “Quantum computing 40 years later” by John Preskill
2. “The Limits of Quantum Computers” by Scott Aaronson
3. “Recent progress in quantum algorithms” by Bacon and van Dam

Whichever article you read, respond to: "Quantum Computing: overrated / underrated?" One paragraph on each viewpoint.

Intermediate-term class plan

Where we are headed in first month

1. Fundamental rules of quantum computing
2. Basic quantum algorithms
3. Programming examples in Google Cirq
4. A NISQ algorithm: quantum approximate optimization algorithm
5. Programming assignment on QAOA in Cirq

Table of contents

Deutsch-Jozsa programs and systems

Algorithm

David Deutsch and Richard Jozsa. Rapid solution of problems by quantum computation. 1992.

Programs

Google Cirq programming example.

Implementation

- Mach-Zehnder interferometer implementation. `https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/SinglePhotonLab/SinglePhotonLab.html`
- Ion trap implementation. Gulde et al. Implementation of the Deutsch–Jozsa algorithm on an ion-trap quantum computer. Letters to Nature. 2003.

Mach-Zehnder interferometer implementation of Deutsch's algorithm

$$|0\rangle \xrightarrow{H} |+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{cases} \xrightarrow{I} |+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} & \xrightarrow{H} |0\rangle \\ \xrightarrow{Z} |-\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} & \xrightarrow{H} |1\rangle \\ \xrightarrow{-Z} -|-\rangle = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} & \xrightarrow{H} -|1\rangle \\ \xrightarrow{-ZZ=-I} -|+\rangle = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} & \xrightarrow{H} -|0\rangle \end{cases}$$

Deutsch-Jozsa algorithm: simplest quantum algorithm showing advantage vs. classical

More formal description

- The 2^n bars are either fake or gold. $f : \{0,1\}^n \to \{0,1\}$.
- Three possibilities:
  1. All are fake. f is constant. $f(x) = 0$ for all $x \in \{0,1\}^n$.
  2. All are gold. f is constant. $f(x) = 1$ for all $x \in \{0,1\}^n$.
  3. Half fake half gold. f is balanced.
     $$\left|\{x \in \{0,1\}^n : f(x) = 0\}\right| = \left|\{x \in \{0,1\}^n : f(x) = 1\}\right| = 2^{n-1}$$
- The oracle U works as follows: $U|c\rangle|t\rangle = |c\rangle|t \oplus f(c)\rangle$
- Try deciding if f is constant or balanced using oracle U only once.

Circuit diagram

Compare to diagram in Rudolph, "Q is for Quantum".

What is in the oracle

For $n = 1$, four possibilities

	f_0	f_1	f_2	f_3
f(0)	0	0	1	1
f(1)	0	1	0	1
	f is constant 0	f is balanced	f is balanced	f is constant 1

Deutsch's algorithm: Deutsch-Jozsa for the $n = 1$ case

Output of circuit is $c = 0$ iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle \otimes |1\rangle = |0\rangle\,|1\rangle = |01\rangle$
2. After first set of Hadamards: $H \otimes H\Big(\,|0\rangle \otimes |1\rangle\,\Big) = H\,|0\rangle \otimes H\,|1\rangle = |+\rangle \otimes |-\rangle =$

$$\left(\frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}\,|0\rangle - \frac{1}{\sqrt{2}}\,|1\rangle\right) = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

From here, let's take an aside via matrix-vector multiplication to build intuition with interference and phase kickback.

Deutsch's algorithm: Deutsch-Jozsa for the $n = 1$ case

Output of circuit is $c = 0$ iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle \otimes |1\rangle = |0\rangle\,|1\rangle = |01\rangle$
2. After first set of Hadamards: $\frac{1}{2}\Big(\,|0\rangle\,\big(\,|0\rangle - |1\rangle\,\big) + |1\rangle\,\big(\,|0\rangle - |1\rangle\,\big)\Big)$
3. After applying oracle U: $U\frac{1}{2}\Big(\,|0\rangle\,\big(\,|0\rangle - |1\rangle\,\big) + |1\rangle\,\big(\,|0\rangle - |1\rangle\,\big)\Big) =$
   $\frac{1}{2}\Big(\,|0\rangle\,\Big(\,|f(0)\rangle - |f(\bar{0})\rangle\,\Big) + |1\rangle\,\Big(\,|f(1)\rangle - |f(\bar{1})\rangle\,\Big)\Big)$
4. This last expression can be factored depending on f:
   $\frac{1}{2}\Big(\,|0\rangle\,\Big(\,|f(0)\rangle - |f(\bar{0})\rangle\,\Big) + |1\rangle\,\Big(\,|f(1)\rangle - |f(\bar{1})\rangle\,\Big)\Big) =$
   $$\begin{cases} \frac{1}{2}\Big(|0\rangle + |1\rangle\Big)\Big(|f(0)\rangle - |f(\bar{0})\rangle\Big) & \text{if } f(0) = f(1) \\ \frac{1}{2}\Big(|0\rangle - |1\rangle\Big)\Big(|f(0)\rangle - |f(\bar{0})\rangle\Big) & \text{if } f(0) \neq f(1) \end{cases} = \begin{cases} |+\rangle\,|-\rangle & \text{if } f(0) = f(1) \\ |-\rangle\,|-\rangle & \text{if } f(0) \neq f(1) \end{cases}$$

The trick where oracle's output on $|t\rangle$ affects phase of $|c\rangle$ is called phase kickback.

Deutsch's algorithm: Deutsch-Jozsa for the $n = 1$ case

Output of circuit is $c = 0$ iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle \otimes |1\rangle = |0\rangle |1\rangle = |01\rangle$
2. After first set of Hadamards: $\frac{1}{2}\Big(|0\rangle \big(|0\rangle - |1\rangle \big) + |1\rangle \big(|0\rangle - |1\rangle \big) \Big)$
3. After applying oracle U:
$$U\frac{1}{2}\Big(|0\rangle \big(|0\rangle - |1\rangle \big) + |1\rangle \big(|0\rangle - |1\rangle \big) \Big) = \begin{cases} |+\rangle |-\rangle & \text{if } f(0) = f(1) \\ |-\rangle |-\rangle & \text{if } f(0) \neq f(1) \end{cases}$$
4. After applying second H on top qubit:
$$\begin{cases} H \otimes I\big(|+\rangle |-\rangle \big) = |0\rangle |-\rangle & \text{if } f(0) = f(1) \\ H \otimes I\big(|-\rangle |-\rangle \big) = |1\rangle |-\rangle & \text{if } f(0) \neq f(1) \end{cases}$$

Table of contents

Deutsch-Jozsa algorithm: pushing Deutsch's algorithm $n > 1$

The Deutch-Jozsa algorithm circuit

Compare to diagram in Rudolph, "Q is for Quantum".

Recall Multiple qubits: the tensor product

Exercise: proof by induction about the Hadamard transform

Show that $|+\rangle^{\otimes n} = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} |c\rangle$

- Base case $n = 1$: $|+\rangle^{\otimes 1} = |+\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle = \frac{1}{2^{1/2}} \sum_{c=0}^{2^1-1} |c\rangle$
- Inductive step assumes statement is true for $n = k-1$, Then for $n = k$:

$$|+\rangle^{\otimes k} = |+\rangle \otimes |+\rangle^{k-1} = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right) \otimes \frac{1}{2^{(k-1)/2}} \sum_{c=0}^{2^{k-1}-1} |c\rangle =$$

$$\frac{1}{2^{1/2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \frac{1}{2^{(k-1)/2}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{2^{k-1}\times 1} = \frac{1}{2^{k/2}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{2^k \times 1}$$

Deutsch's algorithm: Deutsch-Jozsa for the $n = 1$ case

The state after the first set of Hadamards

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle^{\otimes n} \otimes |1\rangle = |0 \ldots 0\rangle\,|1\rangle = |0 \ldots 01\rangle$
2. After first set of Hadamards: $|+\rangle^{\otimes n} \otimes |-\rangle = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} |c\rangle \otimes |-\rangle$

Deutsch's algorithm: Deutsch-Jozsa for the $n = 1$ case

The state after applying oracle U

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle^{\otimes n} \otimes |1\rangle = |0\ldots0\rangle\,|1\rangle = |0\ldots01\rangle$
2. After first set of Hadamards: $|+\rangle^{\otimes n} \otimes |-\rangle = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} |c\rangle \otimes |-\rangle$
3. After applying oracle U:

$$U\Big(\,|+\rangle^{\otimes n} \otimes |-\rangle\,\Big) = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} |c\rangle \otimes \left(\frac{|f(c)\rangle - |\overline{f(c)}\rangle}{\sqrt{2}} \right)$$

$$= \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)}\,|c\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

Lemma: the Hadamard transform

$H^{\otimes n}\left|c\right\rangle = \frac{1}{2^{n/2}}\sum_{m=0}^{2^n-1}(-1)^{c\cdot m}\left|m\right\rangle$

- $$\begin{aligned} &H^{\otimes n}\left|c\right\rangle \\ &= H\left|c_0\right\rangle \otimes H\left|c_1\right\rangle \otimes \ldots \otimes H\left|c_{n-1}\right\rangle \\ &= \frac{1}{\sqrt{2}}\Big(\left|0\right\rangle + (-1)^{c_0}\left|1\right\rangle\Big) \otimes \frac{1}{\sqrt{2}}\Big(\left|0\right\rangle + (-1)^{c_1}\left|1\right\rangle\Big) \otimes \ldots \otimes \frac{1}{\sqrt{2}}\Big(\left|0\right\rangle + (-1)^{c_{n-1}}\left|1\right\rangle\Big) \\ &= \frac{1}{2^{n/2}}\sum_{m=0}^{2^n-1}(-1)^{c_0m_0+c_1m_1+\ldots+c_{n-1}m_{n-1} \mod 2}\left|m\right\rangle \end{aligned}$$
- Try it out for $n = 1$: $H^{\otimes 1}\left|c\right\rangle = \frac{1}{2^{1/2}}\sum_{m=0}^{2^1-1}(-1)^{c\cdot m}\left|m\right\rangle =$ $\frac{1}{\sqrt{2}}(-1)^0\left|0\right\rangle + \frac{1}{\sqrt{2}}(-1)^c\left|1\right\rangle = \begin{cases} \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle = \left|+\right\rangle & \text{if } \left|c\right\rangle = \left|0\right\rangle \\ \frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{1}{\sqrt{2}}\left|1\right\rangle = \left|-\right\rangle & \text{if } \left|c\right\rangle = \left|1\right\rangle \end{cases}$

Deutsch's algorithm: Deutsch-Jozsa for the $n = 1$ case

The state after applying oracle U

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle^{\otimes n} \otimes |1\rangle = |0\dots0\rangle\,|1\rangle = |0\dots01\rangle$
2. After first set of Hadamards: $|+\rangle^{\otimes n} \otimes |-\rangle = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} |c\rangle \otimes |-\rangle$
3. After applying oracle U: $U\Big(|+\rangle^{\otimes n} \otimes |-\rangle\Big) = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$
4. After final set of Hadamards:

$$(H^{\otimes n} \otimes I)\left(\frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\right)$$

$$= \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} \left(\frac{1}{2^{n/2}} \sum_{m=0}^{2^n-1} (-1)^{c\cdot m} |m\rangle\right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{2^n} \sum_{c=0}^{2^n-1} \sum_{m=0}^{2^n-1} (-1)^{f(c)+c\cdot m} |m\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

Deutsch's algorithm: Deutsch-Jozsa for the $n = 1$ case

Output of circuit is 0 iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle^{\otimes n} \otimes |1\rangle = |0\ldots0\rangle\,|1\rangle = |0\ldots01\rangle$
2. After first set of Hadamards: $|+\rangle^{\otimes n} \otimes |-\rangle = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} |c\rangle \otimes |-\rangle$
3. After applying oracle U: $U\Big(|+\rangle^{\otimes n} \otimes |-\rangle\Big) = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$
4. After final set of Hadamards: $(H^{\otimes n} \otimes I)\left(\frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\right) =$
   $\frac{1}{2^n} \sum_{c=0}^{2^n-1} \sum_{m=0}^{2^n-1} (-1)^{f(c)+c\cdot m} |m\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$
5. Amplitude of upper register being $|m\rangle = |0\rangle$:

$$\frac{1}{2^n} \sum_{c=0}^{2^n-1} (-1)^{f(c)}$$

Deutsch's algorithm: Deutsch-Jozsa for the $n = 1$ case

Output of circuit is 0 iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle^{\otimes n} \otimes |1\rangle = |0\dots0\rangle\,|1\rangle = |0\dots01\rangle$
2. After first set of Hadamards: $|+\rangle^{\otimes n} \otimes |-\rangle = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} |c\rangle \otimes |-\rangle$
3. After applying oracle U: $U\left(|+\rangle^{\otimes n} \otimes |-\rangle\right) = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$
4. After final set of Hadamards: $(H^{\otimes n} \otimes I)\left(\frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\right) =$
   $\frac{1}{2^n} \sum_{c=0}^{2^n-1} \sum_{m=0}^{2^n-1} (-1)^{f(c)+c\cdot m} |m\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$
5. Amplitude of upper register being $|m\rangle = |0\rangle$: $\frac{1}{2^n} \sum_{c=0}^{2^n-1} (-1)^{f(c)}$
6. Probability of measuring upper register to get $m = 0$:

$$\left| \frac{1}{2^n} \sum_{c=0}^{2^n-1} (-1)^{f(c)} \right|^2 = \begin{cases} \left|(-1)^{f(c)}\right|^2 = 1 & \text{if } f \text{ is constant} \\ 0 & \text{if } f \text{ is balanced} \end{cases}$$