Quantum computing fundamentals: entanglement

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September 22, 2021

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Announcements

New reading assignment, choose one

1. "Quantum computing 40 years later" by John Preskill
2. "The Limits of Quantum Computers" by Scott Aaronson
3. "Recent progress in quantum algorithms" by Bacon and van Dam

Whichever article you read, respond to: "Quantum Computing: overrated / underrated?" One paragraph on each viewpoint.

Announcements

New programming exercise

https://rutgers.instructure.com/courses/140409/assignments/1660355?module_item_id=5387614

Closing discussion about Deutsch-Jozsa

- How can the oracle be unknown if it has to be a quantum circuit?
- Number of queries needed in classical vs. quantum.
- Takeaway remark about superposition and interference.

Intermediate-term class plan

Where we are headed in first month

1. Fundamentals: superposition / Deutsch-Jozsa
2. Fundamentals: entanglement / Bell inequalities
3. Programming examples in Google Cirq
4. A NISQ algorithm: quantum approximate optimization algorithm
5. Programming assignment on QAOA in Cirq

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Orthogonal measurement bases

Suppose we measured $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$:

$$\text{Measurement result } M_+ = \begin{cases} 0 & \text{with probability } |\frac{1}{\sqrt{2}}|^2 = \frac{1}{2} \\ 1 & \text{with probability } |\frac{1}{\sqrt{2}}|^2 = \frac{1}{2} \end{cases}$$

Suppose we measured $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$:

$$\text{Measurement result } M_- = \begin{cases} 0 & \text{with probability } |\frac{1}{\sqrt{2}}|^2 = \frac{1}{2} \\ 1 & \text{with probability } |\frac{-1}{\sqrt{2}}|^2 = \frac{1}{2} \end{cases}$$

Is there any way to determine if a qubit is either $|+\rangle$ or $|-\rangle$ before its measurement collapse?

Changing bases

Is there any way to determine if a qubit is either $|+\rangle$ or $|-\rangle$ before its measurement collapse?

Yes. Measure in the H basis by applying H prior to measurement.

Holevo bound

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Entangled states: Bell state circuit

Bell state circuit

$$|00\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |\Phi^+\rangle$$

Can $|\Phi^+\rangle$ be treated as the tensor product (composition) of two individual qubits?

Prove that the Bell state cannot be factored into two single-qubit states

Bell state circuit

$$|00\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = |\Phi^+\rangle$$

Can $|\Phi^+\rangle$ be treated as the tensor product (composition) of two individual qubits?
No.

Proof by contradiction:

- Suppose $|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \Big(\alpha|0\rangle + \beta|1\rangle\Big) \otimes \Big(\gamma|0\rangle + \delta|1\rangle\Big) = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$.
- $\alpha\delta = 0$, so either $\alpha = 0$ or $\delta = 0$.
- But $\alpha\gamma = \frac{1}{\sqrt{2}}$, so $\alpha \neq 0$.
- And $\beta\delta = \frac{1}{\sqrt{2}}$, so $\delta \neq 0$ too.

Bell states form an orthogonal basis set

1. $|00\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) = |\Phi^{+}\rangle$
2. $|01\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|01\rangle + |11\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|01\rangle + |10\rangle\Big) = |\Psi^{+}\rangle$
3. $|10\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle - |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle - |11\rangle\Big) = |\Phi^{-}\rangle$
4. $|11\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|01\rangle - |11\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|01\rangle - |10\rangle\Big) = |\Psi^{-}\rangle$

Naive measurement again

Suppose we measured $|\Psi^+\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$:

$$\text{Measurement result } M_{\Psi^+} = \begin{cases} 0,1 & \text{with probability } |\frac{1}{\sqrt{2}}|^2 = \frac{1}{2} \\ 1,0 & \text{with probability } |\frac{1}{\sqrt{2}}|^2 = \frac{1}{2} \end{cases}$$

Suppose we measured $|\Psi^-\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$:

$$\text{Measurement result } M_{\Psi^-} = \begin{cases} 0,1 & \text{with probability } |\frac{1}{\sqrt{2}}|^2 = \frac{1}{2} \\ 1,0 & \text{with probability } |\frac{-1}{\sqrt{2}}|^2 = \frac{1}{2} \end{cases}$$

Is there any way to determine if two qubits are either $|\Psi^+\rangle$ or $|\Psi^-\rangle$ before their measurement collapse?

Changing bases again

Is there any way to determine if two qubits are either $|\Psi^+\rangle$ or $|\Psi^-\rangle$ before their measurement collapse?

Yes. Measure in the Bell basis by applying $CNOT$ and $H \otimes I$ prior to measurement.

1. $|\Phi^+\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle \Big) \xrightarrow{H\otimes I} |00\rangle$
2. $|\Psi^+\rangle = \frac{1}{\sqrt{2}}\Big(|01\rangle + |10\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|01\rangle + |11\rangle \Big) \xrightarrow{H\otimes I} |01\rangle$
3. $|\Phi^-\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle - |11\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle - |10\rangle \Big) \xrightarrow{H\otimes I} |10\rangle$
4. $|\Psi^-\rangle = \frac{1}{\sqrt{2}}\Big(|01\rangle - |10\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|01\rangle - |11\rangle \Big) \xrightarrow{H\otimes I} |11\rangle$

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Superdense coding

Superdense coding circuit

`https://github.com/quantumlib/Cirq/blob/master/examples/superdense_coding.py`

Share Bell pair. Apply operator. Move one qubit. Effectively transmit 2 bits.

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Quantum teleportation

Quantum teleportation circuit

`https://github.com/quantumlib/Cirq/blob/master/examples/quantum_teleportation.py`

Share Bell pair. Entangle local qubit. Measure. Transmit 2 bits. Effectively teleport a qubit.