Quantum computing fundamentals: teleportation and dense coding

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Announcements

First programming exercise

https://rutgers.instructure.com/courses/140409/assignments/1660355?module_item_id=5387614

Intermediate-term class plan

Where we are headed in first month

1. Fundamentals: superposition / Deutsch-Jozsa
2. Fundamentals: entanglement / Bell inequalities
3. Programming examples in Google Cirq
4. A NISQ algorithm: quantum approximate optimization algorithm
5. Programming assignment on QAOA in Cirq

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Entangled states: Bell state circuit

Bell state circuit

$$|00\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = |\Phi^+\rangle$$

Can $|\Phi^+\rangle$ be treated as the tensor product (composition) of two individual qubits?

Prove that the Bell state cannot be factored into two single-qubit states

Bell state circuit

$$|00\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = |\Phi^+\rangle$$

Can $|\Phi^+\rangle$ be treated as the tensor product (composition) of two individual qubits?

No.

Proof by contradiction:

- Suppose $|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \Big(\alpha|0\rangle + \beta|1\rangle\Big) \otimes \Big(\gamma|0\rangle + \delta|1\rangle\Big) = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$.
- $\alpha\delta = 0$, so either $\alpha = 0$ or $\delta = 0$.
- But $\alpha\gamma = \frac{1}{\sqrt{2}}$, so $\alpha \neq 0$.
- And $\beta\delta = \frac{1}{\sqrt{2}}$, so $\delta \neq 0$ too.

Bell states form an orthogonal basis set

1. $|00\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) = |\Phi^+\rangle$
2. $|01\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|01\rangle + |11\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|01\rangle + |10\rangle\Big) = |\Psi^+\rangle$
3. $|10\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle - |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle - |11\rangle\Big) = |\Phi^-\rangle$
4. $|11\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|01\rangle - |11\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|01\rangle - |10\rangle\Big) = |\Psi^-\rangle$

Exercise: orthogonal vectors

- Show that $|\Psi^+\rangle$ and $|\Phi^-\rangle$ are orthogonal.
- Show that $|\Psi^+\rangle$ and $|\Psi^-\rangle$ are orthogonal.

Measurement in standard basis

Suppose we measured $|\Psi^{+}\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$:

$$\text{Measurement result } M_{\Psi^{+}} = \begin{cases} 0,1 & \text{with probability } |\frac{1}{\sqrt{2}}|^2 = \frac{1}{2} \\ 1,0 & \text{with probability } |\frac{1}{\sqrt{2}}|^2 = \frac{1}{2} \end{cases}$$

Suppose we measured $|\Psi^{-}\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$:

$$\text{Measurement result } M_{\Psi^{-}} = \begin{cases} 0,1 & \text{with probability } |\frac{1}{\sqrt{2}}|^2 = \frac{1}{2} \\ 1,0 & \text{with probability } |\frac{-1}{\sqrt{2}}|^2 = \frac{1}{2} \end{cases}$$

Is there any way to determine if two qubits are either $|\Psi^{+}\rangle$ or $|\Psi^{-}\rangle$ before their measurement collapse?

Measurement in Bell basis

Is there any way to determine if two qubits are either $|\Psi^+\rangle$ or $|\Psi^-\rangle$ before their measurement collapse?

Yes. Measure in the Bell basis by applying $CNOT$ and $H \otimes I$ prior to measurement.

1. $|\Phi^+\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle\Big) \xrightarrow{H\otimes I} |00\rangle$
2. $|\Psi^+\rangle = \frac{1}{\sqrt{2}}\Big(|01\rangle + |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|01\rangle + |11\rangle\Big) \xrightarrow{H\otimes I} |01\rangle$
3. $|\Phi^-\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle - |11\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle - |10\rangle\Big) \xrightarrow{H\otimes I} |10\rangle$
4. $|\Psi^-\rangle = \frac{1}{\sqrt{2}}\Big(|01\rangle - |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|01\rangle - |11\rangle\Big) \xrightarrow{H\otimes I} |11\rangle$

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Quantum teleportation

"Teleport" a qubit state by transmitting classical information

1. Alice wishes to give Bob a qubit state $|Q\rangle$.
2. Alice and Bob each have one qubit of a Bell pair in state $|P\rangle$.
3. Alice first entangles $|Q\rangle$ and $|P\rangle$; then, she measures her local two qubits.
4. Alice tells Bob (via classical means) her two-bit measurement result.
5. Bob uses Alice's two bits to perform I, X, Z, or ZX on his qubit to obtain $|Q\rangle$.

Quantum teleportation circuit

`https://github.com/quantumlib/Cirq/blob/master/examples/quantum_teleportation.py`

Quantum teleportation

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Step-by-step qubit state calculation up to Alice's measurement

$$
\begin{aligned}
|Q\rangle \otimes |P\rangle &= \Big(\alpha\,|0\rangle + \beta\,|1\rangle\Big) \otimes \frac{1}{\sqrt{2}}\Big(\,|00\rangle + |11\rangle\,\Big) \\
&= \frac{\alpha}{\sqrt{2}}\,|0\rangle\,\Big(\,|00\rangle + |11\rangle\,\Big) + \frac{\beta}{\sqrt{2}}\,|1\rangle\,\Big(\,|00\rangle + |11\rangle\,\Big) \\
&\xrightarrow{CNOT_{0,1}} \frac{\alpha}{\sqrt{2}}\,|0\rangle\,\Big(\,|00\rangle + |11\rangle\,\Big) + \frac{\beta}{\sqrt{2}}\,|1\rangle\,\Big(\,|10\rangle + |01\rangle\,\Big) \\
&\xrightarrow{H\otimes I\otimes I} \frac{\alpha}{\sqrt{2}}\Big(\,|0\rangle + |1\rangle\,\Big)\Big(\,|00\rangle + |11\rangle\,\Big) + \frac{\beta}{\sqrt{2}}\Big(\,|0\rangle - |1\rangle\,\Big)\Big(\,|10\rangle + |01\rangle\,\Big) \\
&= \frac{1}{2}\,|00\rangle\,\Big(\alpha\,|0\rangle + \beta\,|1\rangle\Big) \\
&\quad + \frac{1}{2}\,|01\rangle\,\Big(\alpha\,|1\rangle + \beta\,|0\rangle\Big) \\
&\quad + \frac{1}{2}\,|10\rangle\,\Big(\alpha\,|0\rangle - \beta\,|1\rangle\Big) \\
&\quad + \frac{1}{2}\,|11\rangle\,\Big(\alpha\,|1\rangle - \beta\,|0\rangle\Big)
\end{aligned}
$$

Quantum teleportation

"Teleport" a qubit state by transmitting classical information

1. Alice wishes to give Bob a qubit state $|Q\rangle$.
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3. Alice first entangles $|Q\rangle$ and $|P\rangle$; then, she measures her local two qubits.
4. Alice tells Bob (via classical means) her two-bit measurement result.
5. Bob uses Alice's two bits to perform I, X, Z, or ZX on his qubit to obtain $|Q\rangle$.

Depending on if Alice measures 00, 01, 10, or 11, Bob applies I, X, Z, or ZX to recover $|Q\rangle$

$$
\begin{aligned}
|Q\rangle \otimes |P\rangle &= \left(\alpha|0\rangle + \beta|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \\
&= \frac{\alpha}{\sqrt{2}}|0\rangle\left(|00\rangle + |11\rangle\right) + \frac{\beta}{\sqrt{2}}|1\rangle\left(|00\rangle + |11\rangle\right) \\
&\xrightarrow{CNOT_{0,1}} \frac{\alpha}{\sqrt{2}}|0\rangle\left(|00\rangle + |11\rangle\right) + \frac{\beta}{\sqrt{2}}|1\rangle\left(|10\rangle + |01\rangle\right) \\
&\xrightarrow{H\otimes I\otimes I} \frac{\alpha}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\left(|00\rangle + |11\rangle\right) + \frac{\beta}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\left(|10\rangle + |01\rangle\right) \\
&= \frac{1}{2}|00\rangle\left(\alpha|0\rangle + \beta|1\rangle\right) \quad \text{Alice measures 00 so Bob applies I} \\
&+ \frac{1}{2}|01\rangle\left(\alpha|1\rangle + \beta|0\rangle\right) \quad \text{Alice measures 01 so Bob applies X} \\
&+ \frac{1}{2}|10\rangle\left(\alpha|0\rangle - \beta|1\rangle\right) \quad \text{Alice measures 10 so Bob applies Z} \\
&+ \frac{1}{2}|11\rangle\left(\alpha|1\rangle - \beta|0\rangle\right) \quad \text{Alice measures 11 so Bob applies ZX}
\end{aligned}
$$

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Superdense coding

Transmit 2 bits of classical information by sending 1 qubit

1. Alice wishes to tell Bob two bits of information: 00, 01, 10, or 11.
2. Alice and Bob each have one qubit of a Bell pair in state $|P\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$.
3. Alice performs I, X, Z, or ZX on her qubit; she then sends her qubit to Bob.
4. Bob measures in the Bell basis to receive 00, 01, 10, or 11.

Superdense coding circuit

`https://github.com/quantumlib/Cirq/blob/master/examples/superdense_coding.py`

Superdense coding

Transmit 2 bits of classical information by sending 1 qubit

1. Alice wishes to tell Bob two bits of information: 00, 01, 10, or 11.
2. Alice and Bob each have one qubit of a Bell pair in state $|P\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big)$.
3. Alice performs I, X, Z, or ZX on her qubit; she then sends her qubit to Bob.
4. Bob measures in the Bell basis to receive 00, 01, 10, or 11.

Alice applies different operators on her qubit so Bob measures the message

1. $|P\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) \xrightarrow{I\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle \Big) \xrightarrow{H\otimes I} |00\rangle$
2. $|P\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) \xrightarrow{X\otimes I} \frac{1}{\sqrt{2}}\Big(|10\rangle + |01\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|11\rangle + |01\rangle \Big) \xrightarrow{H\otimes I} |01\rangle$
3. $|P\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) \xrightarrow{Z\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle - |11\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle - |10\rangle \Big) \xrightarrow{H\otimes I} |10\rangle$
4. $|P\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) \xrightarrow{ZX\otimes I} \frac{1}{\sqrt{2}}\Big(- |10\rangle + |01\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(- |11\rangle + |01\rangle \Big) \xrightarrow{H\otimes I} |11\rangle$

Channel capacity of dense coding

How does dense coding scale?

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Dense coding with multipartite qubit entanglement

3 qubit maximally entangled orthogonal basis set

1. $|000\rangle \xrightarrow{H\otimes I\otimes I} \frac{1}{\sqrt{2}}\Big(|000\rangle + |100\rangle\Big) \xrightarrow{CNOT_{0,1},CNOT_{0,2}} \frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big)$
2. $|001\rangle \xrightarrow{H\otimes I\otimes I} \frac{1}{\sqrt{2}}\Big(|001\rangle + |101\rangle\Big) \xrightarrow{CNOT_{0,1},CNOT_{0,2}} \frac{1}{\sqrt{2}}\Big(|001\rangle + |110\rangle\Big)$
3. $|010\rangle \xrightarrow{H\otimes I\otimes I} \frac{1}{\sqrt{2}}\Big(|010\rangle + |110\rangle\Big) \xrightarrow{CNOT_{0,1},CNOT_{0,2}} \frac{1}{\sqrt{2}}\Big(|010\rangle + |101\rangle\Big)$
4. $|011\rangle \xrightarrow{H\otimes I\otimes I} \frac{1}{\sqrt{2}}\Big(|011\rangle + |111\rangle\Big) \xrightarrow{CNOT_{0,1},CNOT_{0,2}} \frac{1}{\sqrt{2}}\Big(|011\rangle + |100\rangle\Big)$
5. $|100\rangle \xrightarrow{H\otimes I\otimes I} \frac{1}{\sqrt{2}}\Big(|000\rangle - |100\rangle\Big) \xrightarrow{CNOT_{0,1},CNOT_{0,2}} \frac{1}{\sqrt{2}}\Big(|000\rangle - |111\rangle\Big)$
6. $|101\rangle \xrightarrow{H\otimes I\otimes I} \frac{1}{\sqrt{2}}\Big(|001\rangle - |101\rangle\Big) \xrightarrow{CNOT_{0,1},CNOT_{0,2}} \frac{1}{\sqrt{2}}\Big(|001\rangle - |110\rangle\Big)$
7. $|110\rangle \xrightarrow{H\otimes I\otimes I} \frac{1}{\sqrt{2}}\Big(|010\rangle - |110\rangle\Big) \xrightarrow{CNOT_{0,1},CNOT_{0,2}} \frac{1}{\sqrt{2}}\Big(|010\rangle - |101\rangle\Big)$
8. $|111\rangle \xrightarrow{H\otimes I\otimes I} \frac{1}{\sqrt{2}}\Big(|011\rangle - |111\rangle\Big) \xrightarrow{CNOT_{0,1},CNOT_{0,2}} \frac{1}{\sqrt{2}}\Big(|011\rangle - |100\rangle\Big)$

Exercise: show that this is an orthogonal set of vectors.

Transmit 3 bits of classical information by sending 2 qubits

Alice applies different operators on her qubits so Bob measures the message

1. $\frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big) \xrightarrow{I\otimes I\otimes I} \frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big) \xrightarrow{CNOT_{0,2}, CNOT_{0,1}} \frac{1}{\sqrt{2}}\Big(|000\rangle + |100\rangle\Big) \xrightarrow{H\otimes I\otimes I} |000\rangle$
2. $\frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big) \xrightarrow{X\otimes X\otimes I} \frac{1}{\sqrt{2}}\Big(|110\rangle + |001\rangle\Big) \xrightarrow{CNOT_{0,2}, CNOT_{0,1}} \frac{1}{\sqrt{2}}\Big(|101\rangle + |001\rangle\Big) \xrightarrow{H\otimes I\otimes I} |001\rangle$
3. $\frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big) \xrightarrow{I\otimes X\otimes I} \frac{1}{\sqrt{2}}\Big(|010\rangle + |101\rangle\Big) \xrightarrow{CNOT_{0,2}, CNOT_{0,1}} \frac{1}{\sqrt{2}}\Big(|010\rangle + |110\rangle\Big) \xrightarrow{H\otimes I\otimes I} |010\rangle$
4. $\frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big) \xrightarrow{X\otimes I\otimes I} \frac{1}{\sqrt{2}}\Big(|100\rangle + |011\rangle\Big) \xrightarrow{CNOT_{0,2}, CNOT_{0,1}} \frac{1}{\sqrt{2}}\Big(|111\rangle + |011\rangle\Big) \xrightarrow{H\otimes I\otimes I} |011\rangle$
5. $\frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big) \xrightarrow{Z\otimes I\otimes I} \frac{1}{\sqrt{2}}\Big(|000\rangle - |111\rangle\Big) \xrightarrow{CNOT_{0,2}, CNOT_{0,1}} \frac{1}{\sqrt{2}}\Big(|000\rangle - |100\rangle\Big) \xrightarrow{H\otimes I\otimes I} |100\rangle$
6. $\frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big) \xrightarrow{ZX\otimes X\otimes I} \frac{1}{\sqrt{2}}\Big(-|110\rangle + |001\rangle\Big) \xrightarrow{CNOT_{0,2}, CNOT_{0,1}} \frac{1}{\sqrt{2}}\Big(-|101\rangle + |001\rangle\Big) \xrightarrow{H\otimes I\otimes I} |101\rangle$
7. $\frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big) \xrightarrow{Z\otimes X\otimes I} \frac{1}{\sqrt{2}}\Big(|010\rangle - |101\rangle\Big) \xrightarrow{CNOT_{0,2}, CNOT_{0,1}} \frac{1}{\sqrt{2}}\Big(|010\rangle - |110\rangle\Big) \xrightarrow{H\otimes I\otimes I} |110\rangle$
8. $\frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big) \xrightarrow{ZX\otimes I\otimes I} \frac{1}{\sqrt{2}}\Big(-|100\rangle + |011\rangle\Big) \xrightarrow{CNOT_{0,2}, CNOT_{0,1}} \frac{1}{\sqrt{2}}\Big(-|111\rangle + |011\rangle\Big) \xrightarrow{H\otimes I\otimes I} |111\rangle$

Transmit 4 bits of classical information by sending 2 qubits

4 qubit maximally entangled orthogonal basis set

Gustavo Rigolin. Superdense coding using multipartite states. arXiv. 2004.
`https://arxiv.org/pdf/quant-ph/0407193.pdf`

Dense coding with multipartite qubit entanglement

Maximum dense coding channel capacity of 2 bits received per *qubit* sent.

Can one do better if using something other than *qubits*?

Hu et al.. Beating the channel capacity limit for superdense coding with entangled ququarts. Science Advances. 2018.
`https://www.science.org/doi/10.1126/sciadv.aat9304`

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Dense coding with high-dimensional qudit entanglement

Qudits

	1 qudit basis vectors	2 qudit basis vectors	3 qudit basis vectors
Qubit	$\|0\rangle,\|1\rangle$	$\|00\rangle,\|01\rangle,\|10\rangle,\|11\rangle$	$\|000\rangle,\|001\rangle,\|010\rangle,...,\|111\rangle$
Qutrit	$\|0\rangle,\|1\rangle,\|2\rangle$	$\|00\rangle,\|01\rangle,\|02\rangle,\|10\rangle,...,\|22\rangle$	$\|000\rangle,...,\|002\rangle,\|010\rangle,...,\|222\rangle$
Ququart	$\|0\rangle,\|1\rangle,\|2\rangle,\|3\rangle$	$\|00\rangle,...,\|03\rangle,\|10\rangle,...,\|33\rangle$	$\|000\rangle,...,\|003\rangle,\|010\rangle,...,\|333\rangle$

So for example, 2 qutrits with an orthogonal state space size of $3^2 = 9$ can encode the state space of 3 qubits (which has state space size of $2^3 = 8$).

DFT matrix: generalized Hadamard for D-dimensional qudits

$$W = \frac{1}{\sqrt{D}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{D-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(D-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{D-1} & \omega^{2(D-1)} & \cdots & \omega^{(D-1)(D-1)} \end{bmatrix}$$

Where

$$\omega = e^{\frac{2}{D}\pi i}$$

And recall that

$$e^{ix} = \cos x + i \sin x$$

Teo Banica. Complex Hadamard matrices and applications. 2021. hal-02317067v2
`https://arxiv.org/abs/1910.06911`

DFT matrix: generalized Hadamard for D-dimensional qudits

$D = 2$, Hadamard matrix for qubits

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & \omega^1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & e^{\frac{2}{2}\pi i} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$D = 3$, Hadamard matrix for qutrits

$$W = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^1 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2}{3}\pi i} & e^{\frac{4}{3}\pi i} \\ 1 & e^{\frac{4}{3}\pi i} & e^{\frac{8}{3}\pi i} \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2}{3}\pi i} & e^{\frac{-2}{3}\pi i} \\ 1 & e^{\frac{-2}{3}\pi i} & e^{\frac{2}{3}\pi i} \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} + \frac{\sqrt{3}}{2}i & -\frac{1}{2} - \frac{\sqrt{3}}{2}i \\ 1 & -\frac{1}{2} - \frac{\sqrt{3}}{2}i & -\frac{1}{2} + \frac{\sqrt{3}}{2}i \end{bmatrix}$$

Yurtalan et al.. Implementation of a Walsh-Hadamard Gate in a Superconducting Qutrit. PRL. 2020. `https://journals.aps.org/prl/`

Generalized CNOT gate

$D = 2$, CNOT for qubits

$$|x, y\rangle \xrightarrow{CNOT} |x, x + y \mod 2\rangle$$

1. $|0, 0\rangle \xrightarrow{CNOT} |0, 0\rangle$
2. $|0, 1\rangle \xrightarrow{CNOT} |0, 1\rangle$
3. $|1, 0\rangle \xrightarrow{CNOT} |1, 1\rangle$
4. $|1, 1\rangle \xrightarrow{CNOT} |1, 0\rangle$

$D = 3$, CPLUS for qutrits

$$|x, y\rangle \xrightarrow{CPLUS} |x, x + y \mod 3\rangle$$

1. $|0, 0\rangle \xrightarrow{CPLUS} |0, 0\rangle$
2. $|0, 1\rangle \xrightarrow{CPLUS} |0, 1\rangle$
3. $|0, 2\rangle \xrightarrow{CPLUS} |0, 2\rangle$
4. $|1, 0\rangle \xrightarrow{CPLUS} |1, 1\rangle$
5. $|1, 1\rangle \xrightarrow{CPLUS} |1, 2\rangle$
6. $|1, 2\rangle \xrightarrow{CPLUS} |1, 0\rangle$
7. $|2, 0\rangle \xrightarrow{CPLUS} |2, 2\rangle$
8. $|2, 1\rangle \xrightarrow{CPLUS} |2, 0\rangle$
9. $|2, 2\rangle \xrightarrow{CPLUS} |2, 1\rangle$

Çorbaci et al.. Construction of two qutrit entanglement by using magnetic resonance selective pulse sequences. Journal of Physics: Conference Series. 2016. `https://iopscience.iop.org/article/10.1088/1742-6596/766/1/012014`

Dense coding with high-dimensional qudit entanglement

2 qutrit maximally entangled orthogonal basis set

1. $|00\rangle \xrightarrow{W\otimes I} \frac{1}{\sqrt{3}}\Big(|00\rangle + |10\rangle + |20\rangle\Big) \xrightarrow{CPLUS} \frac{1}{\sqrt{3}}\Big(|00\rangle + |11\rangle + |22\rangle\Big)$
2. $|01\rangle \xrightarrow{W\otimes I} \frac{1}{\sqrt{3}}\Big(|01\rangle + |11\rangle + |21\rangle\Big) \xrightarrow{CPLUS} \frac{1}{\sqrt{3}}\Big(|01\rangle + |12\rangle + |20\rangle\Big)$
3. $|02\rangle \xrightarrow{W\otimes I} \frac{1}{\sqrt{3}}\Big(|02\rangle + |12\rangle + |22\rangle\Big) \xrightarrow{CPLUS} \frac{1}{\sqrt{3}}\Big(|02\rangle + |10\rangle + |21\rangle\Big)$
4. $|10\rangle \xrightarrow{W\otimes I} \frac{1}{\sqrt{3}}\Big(|00\rangle + \omega\,|10\rangle + \omega^2\,|20\rangle\Big) \xrightarrow{CPLUS} \frac{1}{\sqrt{3}}\Big(|00\rangle + \omega\,|11\rangle + \omega^2\,|22\rangle\Big)$
5. $|11\rangle \xrightarrow{W\otimes I} \frac{1}{\sqrt{3}}\Big(|01\rangle + \omega\,|11\rangle + \omega^2\,|21\rangle\Big) \xrightarrow{CPLUS} \frac{1}{\sqrt{3}}\Big(|01\rangle + \omega\,|12\rangle + \omega^2\,|20\rangle\Big)$
6. $|12\rangle \xrightarrow{W\otimes I} \frac{1}{\sqrt{3}}\Big(|02\rangle + \omega\,|12\rangle + \omega^2\,|22\rangle\Big) \xrightarrow{CPLUS} \frac{1}{\sqrt{3}}\Big(|02\rangle + \omega\,|10\rangle + \omega^2\,|21\rangle\Big)$
7. $|20\rangle \xrightarrow{W\otimes I} \frac{1}{\sqrt{3}}\Big(|00\rangle + \omega^2\,|10\rangle + \omega\,|20\rangle\Big) \xrightarrow{CPLUS} \frac{1}{\sqrt{3}}\Big(|00\rangle + \omega^2\,|11\rangle + \omega\,|22\rangle\Big)$
8. $|21\rangle \xrightarrow{W\otimes I} \frac{1}{\sqrt{3}}\Big(|01\rangle + \omega^2\,|11\rangle + \omega\,|21\rangle\Big) \xrightarrow{CPLUS} \frac{1}{\sqrt{3}}\Big(|01\rangle + \omega^2\,|12\rangle + \omega\,|20\rangle\Big)$
9. $|22\rangle \xrightarrow{W\otimes I} \frac{1}{\sqrt{3}}\Big(|02\rangle + \omega^2\,|12\rangle + \omega\,|22\rangle\Big) \xrightarrow{CPLUS} \frac{1}{\sqrt{3}}\Big(|02\rangle + \omega^2\,|10\rangle + \omega\,|21\rangle\Big)$

Exercise: show that this is an orthogonal set of vectors.

Focus on lines 4 and 7. Recall complex vector dot product.

Breaking the binary abstraction

Is ternary (and beyond) logic useful in classical computing?

Is ternary (and beyond) logic useful in quantum computing?

- PRO: higher quantum channel capacity.
- CON: qubits are much easier to engineer that qutrits.

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Dense coding programs and systems

In this class, we study quantum algorithms down to hardware

- `https://quantumai.google/cirq/qudits`
- `https://qiskit.org/textbook/ch-quantum-hardware/accessing_higher_energy_states.html`
- Cervera-Lierta. Experimental high-dimensional Greenberger-Horne-Zeilinger entanglement with superconducting transmon qutrits. arXiv. 2021. `https://arxiv.org/abs/2104.05627`

Challenge programming assignment: qutrit dense coding in Cirq

Can substitute either programming assignment on QAOA or VQE.

Timely research project: world-first demo of qutrit dense coding on IBM Q

Likely key challenge: qubits currently engineered for CNOTs.