Quantum computing fundamentals: Hardy's paradox / CHSH / Bell's inequality

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Intermediate-term class plan

Where we are headed in first month

1. Fundamentals: superposition / Deutsch-Jozsa
2. Fundamentals: entanglement / Bell inequalities
3. Programming examples in Google Cirq
4. Shor's algorithm (new)
5. A NISQ algorithm: quantum approximate optimization algorithm
6. Programming assignment on QAOA in Cirq

Longer-term class plan

The remaining two months

1. Systems view of quantum computing
2. Programming abstractions: stabilizers, tensor networks
3. VQE: quantum chemistry
4. Quantum error correction codes (new)
5. Quantum architecture and micro-architecture
6. Prototype devices: superconductors and ion traps

What is being cut

1. 2020 quantum advantage debates
2. Student presentations: $2 \to 1$

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Dense coding with high-dimensional qudit entanglement

Laser optics physical experiment

Hu et al.. Beating the channel capacity limit for superdense coding with entangled ququarts. Science Advances. 2018.
`https://www.science.org/doi/10.1126/sciadv.aat9304`

Challenge programming assignment: qutrit dense coding in Cirq

If you take on building this prototype, can substitute either programming assignment on QAOA or VQE.

Timely research project: world-first demo of qutrit dense coding on IBM Q

Likely key challenge: qubits currently engineered for CNOTs.

Dense coding with high-dimensional qudit entanglement

Qudits

	1 qudit basis vectors	2 qudit basis vectors	3 qudit basis vectors
Qubit	$\|0\rangle,\|1\rangle$	$\|00\rangle,\|01\rangle,\|10\rangle,\|11\rangle$	$\|000\rangle,\|001\rangle,\|010\rangle,...,\|111\rangle$
Qutrit	$\|0\rangle,\|1\rangle,\|2\rangle$	$\|00\rangle,\|01\rangle,\|02\rangle,\|10\rangle,...,\|22\rangle$	$\|000\rangle,...,\|002\rangle,\|010\rangle,...,\|222\rangle$
Ququart	$\|0\rangle,\|1\rangle,\|2\rangle,\|3\rangle$	$\|00\rangle,...,\|03\rangle,\|10\rangle,...,\|33\rangle$	$\|000\rangle,...,\|003\rangle,\|010\rangle,...,\|333\rangle$

So for example, 2 qutrits with an orthogonal state space size of $3^2 = 9$ can encode the state space of 3 qubits (which has state space size of $2^3 = 8$).

DFT matrix: generalized Hadamard for D-dimensional qudits

$$W = \frac{1}{\sqrt{D}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{D-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(D-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{D-1} & \omega^{2(D-1)} & \cdots & \omega^{(D-1)(D-1)} \end{bmatrix}$$

Where

$$\omega = e^{\frac{2}{D}\pi i}$$

And recall that

$$e^{ix} = \cos x + i \sin x$$

Teo Banica. Complex Hadamard matrices and applications. 2021. hal-02317067v2
`https://arxiv.org/abs/1910.06911`

DFT matrix: generalized Hadamard for D-dimensional qudits

$D = 2$, Hadamard matrix for qubits

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & \omega^1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & e^{\frac{2}{2}\pi i} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$D = 3$, Hadamard matrix for qutrits

$$W = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^1 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2}{3}\pi i} & e^{\frac{4}{3}\pi i} \\ 1 & e^{\frac{4}{3}\pi i} & e^{\frac{8}{3}\pi i} \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2}{3}\pi i} & e^{\frac{-2}{3}\pi i} \\ 1 & e^{\frac{-2}{3}\pi i} & e^{\frac{2}{3}\pi i} \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} + \frac{\sqrt{3}}{2} i & -\frac{1}{2} - \frac{\sqrt{3}}{2} i \\ 1 & -\frac{1}{2} - \frac{\sqrt{3}}{2} i & -\frac{1}{2} + \frac{\sqrt{3}}{2} i \end{bmatrix}$$

Yurtalan et al.. Implementation of a Walsh-Hadamard Gate in a Superconducting Qutrit. PRL. 2020. https://journals.aps.org/prl/

Generalized CNOT gate

$D = 2$, CNOT for qubits

$$|x, y\rangle \xrightarrow{CNOT} |x, x + y \mod 2\rangle$$

1. $|0, 0\rangle \xrightarrow{CNOT} |0, 0\rangle$
2. $|0, 1\rangle \xrightarrow{CNOT} |0, 1\rangle$
3. $|1, 0\rangle \xrightarrow{CNOT} |1, 1\rangle$
4. $|1, 1\rangle \xrightarrow{CNOT} |1, 0\rangle$

$D = 3$, CPLUS for qutrits

$$|x, y\rangle \xrightarrow{CPLUS} |x, x + y \mod 3\rangle$$

1. $|0, 0\rangle \xrightarrow{CPLUS} |0, 0\rangle$
2. $|0, 1\rangle \xrightarrow{CPLUS} |0, 1\rangle$
3. $|0, 2\rangle \xrightarrow{CPLUS} |0, 2\rangle$
4. $|1, 0\rangle \xrightarrow{CPLUS} |1, 1\rangle$
5. $|1, 1\rangle \xrightarrow{CPLUS} |1, 2\rangle$
6. $|1, 2\rangle \xrightarrow{CPLUS} |1, 0\rangle$
7. $|2, 0\rangle \xrightarrow{CPLUS} |2, 2\rangle$
8. $|2, 1\rangle \xrightarrow{CPLUS} |2, 0\rangle$
9. $|2, 2\rangle \xrightarrow{CPLUS} |2, 1\rangle$

Çorbaci et al.. Construction of two qutrit entanglement by using magnetic resonance selective pulse sequences. Journal of Physics: Conference Series. 2016. `https://iopscience.iop.org/article/10.1088/1742-6596/766/1/012014`

Dense coding with high-dimensional qudit entanglement

2 qutrit maximally entangled orthogonal basis set

1. $|00\rangle \xrightarrow{W\otimes I} \frac{1}{\sqrt{3}}\Big(|00\rangle + |10\rangle + |20\rangle\Big) \xrightarrow{CPLUS} \frac{1}{\sqrt{3}}\Big(|00\rangle + |11\rangle + |22\rangle\Big)$
2. $|01\rangle \xrightarrow{W\otimes I} \frac{1}{\sqrt{3}}\Big(|01\rangle + |11\rangle + |21\rangle\Big) \xrightarrow{CPLUS} \frac{1}{\sqrt{3}}\Big(|01\rangle + |12\rangle + |20\rangle\Big)$
3. $|02\rangle \xrightarrow{W\otimes I} \frac{1}{\sqrt{3}}\Big(|02\rangle + |12\rangle + |22\rangle\Big) \xrightarrow{CPLUS} \frac{1}{\sqrt{3}}\Big(|02\rangle + |10\rangle + |21\rangle\Big)$
4. $|10\rangle \xrightarrow{W\otimes I} \frac{1}{\sqrt{3}}\Big(|00\rangle + \omega\,|10\rangle + \omega^2\,|20\rangle\Big) \xrightarrow{CPLUS} \frac{1}{\sqrt{3}}\Big(|00\rangle + \omega\,|11\rangle + \omega^2\,|22\rangle\Big)$
5. $|11\rangle \xrightarrow{W\otimes I} \frac{1}{\sqrt{3}}\Big(|01\rangle + \omega\,|11\rangle + \omega^2\,|21\rangle\Big) \xrightarrow{CPLUS} \frac{1}{\sqrt{3}}\Big(|01\rangle + \omega\,|12\rangle + \omega^2\,|20\rangle\Big)$
6. $|12\rangle \xrightarrow{W\otimes I} \frac{1}{\sqrt{3}}\Big(|02\rangle + \omega\,|12\rangle + \omega^2\,|22\rangle\Big) \xrightarrow{CPLUS} \frac{1}{\sqrt{3}}\Big(|02\rangle + \omega\,|10\rangle + \omega^2\,|21\rangle\Big)$
7. $|20\rangle \xrightarrow{W\otimes I} \frac{1}{\sqrt{3}}\Big(|00\rangle + \omega^2\,|10\rangle + \omega\,|20\rangle\Big) \xrightarrow{CPLUS} \frac{1}{\sqrt{3}}\Big(|00\rangle + \omega^2\,|11\rangle + \omega\,|22\rangle\Big)$
8. $|21\rangle \xrightarrow{W\otimes I} \frac{1}{\sqrt{3}}\Big(|01\rangle + \omega^2\,|11\rangle + \omega\,|21\rangle\Big) \xrightarrow{CPLUS} \frac{1}{\sqrt{3}}\Big(|01\rangle + \omega^2\,|12\rangle + \omega\,|20\rangle\Big)$
9. $|22\rangle \xrightarrow{W\otimes I} \frac{1}{\sqrt{3}}\Big(|02\rangle + \omega^2\,|12\rangle + \omega\,|22\rangle\Big) \xrightarrow{CPLUS} \frac{1}{\sqrt{3}}\Big(|02\rangle + \omega^2\,|10\rangle + \omega\,|21\rangle\Big)$

Exercise: show that this is an orthogonal set of vectors.

Focus on lines 4 and 7. Recall complex vector dot product.

Transmit 2 trits of classical information by sending 1 qutrit

Alice applies different operators on her qutrits so Bob measures the message

1. $\frac{1}{\sqrt{3}}\Big(|00\rangle + |11\rangle + |22\rangle\Big) \xrightarrow{I \otimes I} \frac{1}{\sqrt{3}}\Big(|00\rangle + |11\rangle + |22\rangle\Big) \xrightarrow{CMINUS_{0,1}} \frac{1}{\sqrt{3}}\Big(|00\rangle + |10\rangle + |20\rangle\Big) \xrightarrow{W^{-1} \otimes I} |00\rangle$
2. $\ldots$

See the set of 9 Weyl operators in `https://iopscience.iop.org/article/10.1088/1742-6596/766/1/012014/pdf`

Breaking the binary abstraction

Is ternary (and beyond) logic useful in classical computing?

- Historically was useful in analog computing.
- Drive to decrease supply voltage while increasing noise resilience made processing in binary the dominant paradigm.
- Current still has uses in storage, at least.

Is ternary (and beyond) logic useful in quantum computing?

- PRO: higher quantum channel capacity.
- CON: qubits are much easier to engineer than qutrits.

Dense coding programs and systems

In this class, we study quantum algorithms down to hardware

- `https://quantumai.google/cirq/qudits`
- `https://qiskit.org/textbook/ch-quantum-hardware/accessing_higher_energy_states.html`
- Cervera-Lierta. Experimental high-dimensional Greenberger-Horne-Zeilinger entanglement with superconducting transmon qutrits. arXiv. 2021. `https://arxiv.org/abs/2104.05627`

Challenge programming assignment: qutrit dense coding in Cirq

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The lens we take when studying quantum computing

- When quantum physics was first discovered, the mathematics of entanglement led to shocking conclusions.
- If you can keep systems coherent (isolated), they can exhibit superposition and entanglement.
- Einstein and others: there shouldn't be "spooky action at a distance" so there must be some local hidden-variable. The task was then to prove or disprove local hidden-variables.
- But protocols and experiments like Hardy's, GHZ, CHSH, and Aspect experimentally rejected local hidden-variable theory.

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Test of entanglement

Two isolated parties Alice and Bob

- Alice gets coin toss x, replies a
- Bob gets coin toss y, replies b

Goal: maximize $a \oplus b = x \wedge y$

x	y	$x \wedge y$	$a \oplus b$	winning options for (a, b)
0	0	0	0	(0,0) or (1,1)
0	1	0	0	(0,0) or (1,1)
1	0	0	0	(0,0) or (1,1)
1	1	1	1	(0,1) or (1,0)

Best classical strategy to maximize $a \oplus b = x \wedge y$

Proof that any assignment to a and b cannot always satisfy $a \oplus b = x \wedge y$

1. Let a_0 be Alice's response if she sees $x = 0$
2. Let a_1 be Alice's response if she sees $x = 1$
3. Let b_0 be Bob's response if she sees $y = 0$
4. Let b_1 be Bob's response if she sees $y = 1$

Satisfy $a \oplus b = x \wedge y$

1. $a_0 \oplus b_0 = 0$
2. $a_0 \oplus b_1 = 0$
3. $a_1 \oplus b_0 = 0$
4. $a_1 \oplus b_1 = 1$

Sum (mod 2) of left side

$(a_0 \oplus b_0) \oplus (a_0 \oplus b_1) \oplus (a_1 \oplus b_0) \oplus (a_1 \oplus b_1) =$
$(a_0 \oplus a_0) \oplus (a_1 \oplus a_1) \oplus (b_0 \oplus b_0) \oplus (b_1 \oplus b_1) = 0$

Sum (mod 2) of right side

1

Best classical strategy to maximize $a \oplus b = x \wedge y$

Even if the two shared randomness, the random coin toss of x and y prevents use of shared randomness.
Best you can do is 3/4. Give a couple ways of getting 3/4

A quantum strategy to maximize $a \oplus b = x \wedge y$

Alice and Bob share entangled pair $|\Phi\rangle$

$|\Phi\rangle = \frac{1}{\sqrt{12}}\Big(3\,|00\rangle + |01\rangle + |10\rangle - |11\rangle\Big)$

$(x, y) = (0, 0)$ So Alice and Bob both apply I:

$$(I \otimes I)\,|\Phi\rangle = \frac{1}{\sqrt{12}}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Measurement yields

$$\begin{cases} (a, b) = (0, 0), \text{ a win, with probability } \frac{9}{12} \\ (a, b) = (0, 1), \text{ a loss, with probability } \frac{1}{12} \\ (a, b) = (1, 0), \text{ a loss, with probability } \frac{1}{12} \\ (a, b) = (1, 1), \text{ a win, with probability } \frac{1}{12} \end{cases}$$

A quantum strategy to maximize $a \oplus b = x \wedge y$

Alice and Bob share entangled pair $|\Phi\rangle$

$$|\Phi\rangle = \frac{1}{\sqrt{12}}\Big(3|00\rangle + |01\rangle + |10\rangle - |11\rangle\Big)$$

$(x, y) = (0, 1)$ So Alice applies I, Bob applies H:

$$(I \otimes H)|\Phi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}\frac{1}{\sqrt{12}}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2\sqrt{6}}\begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

Measurement yields

$$\begin{cases} (a, b) = (0, 0), \text{ a win, with probability } \frac{4}{6} \\ (a, b) = (0, 1), \text{ a loss, with probability } \frac{1}{6} \\ (a, b) = (1, 0), \text{ a loss, with probability } 0 \\ (a, b) = (1, 1), \text{ a win, with probability } \frac{1}{6} \end{cases}$$

A quantum strategy to maximize $a \oplus b = x \wedge y$

Alice and Bob share entangled pair $|\Phi\rangle$

$$|\Phi\rangle = \frac{1}{\sqrt{12}}\Big(3\,|00\rangle + |01\rangle + |10\rangle - |11\rangle\Big)$$

$(x, y) = (1, 0)$ So Alice applies H, Bob applies I:

$$(H \otimes I)\,|\Phi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \frac{1}{\sqrt{12}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2\sqrt{6}} \begin{bmatrix} 4 \\ 0 \\ 2 \\ 2 \end{bmatrix}$$

Measurement yields

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A quantum strategy to maximize $a \oplus b = x \wedge y$

Alice and Bob share entangled pair $|\Phi\rangle$

$|\Phi\rangle = \frac{1}{\sqrt{12}}\Big(3\,|00\rangle + |01\rangle + |10\rangle - |11\rangle\Big)$

$(x, y) = (1, 1)$ So Alice and Bob both apply H:

$$(H \otimes H)\,|\Phi\rangle = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \frac{1}{\sqrt{12}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{4\sqrt{3}} \begin{bmatrix} 4 \\ 4 \\ 4 \\ 0 \end{bmatrix}$$

Measurement yields

$$\begin{cases} (a, b) = (0, 0), \text{ a loss, with probability } \frac{1}{3} \\ (a, b) = (0, 1), \text{ a win, with probability } \frac{1}{3} \\ (a, b) = (1, 0), \text{ a win, with probability } \frac{1}{3} \\ (a, b) = (1, 1), \text{ a loss, with probability } 0 \end{cases}$$

The lens we take when studying quantum computing

Cannot have both locality and realism

- Locality: “means that information and causation act locally, not faster than light”
- Realism: “means that physical systems have definite, well-defined properties (even if those properties may be unknown to us)”

Source: de Wolf. Quantum Computing: Lecture Notes

Unpalatable choices

- Keep locality and sacrifice realism: no definite narrative of the world
- Keep realism and sacrifice locality: spooky-action-at-a-distance