# Tensor Networks

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# Long-range class plan

| Date | Class topic | Readings and assignments |
|---|---|---|
| 10/20 | NISQ algorithms: QAOA | |
| 10/25 | NISQ algorithms: QAOA | QAOA lab out |
| 10/27 | Quantum computing: systems view | New reading assignment release |
| 11/1 | Languages: stabilizers | |
| 11/3 | Languages: tensor networks | |
| 11/8 | Languages: density matrices, noise | QAOA lab part 1 due |
| 11/10 | Languages: logical abstractions | |
| 11/15 | Quantum error correction codes | |
| 11/17 | NISQ algorithms: quantum chemistry | |
| 11/22 | NISQ algorithms: VQE | QAOA lab all due, VQE lab out |
| 11/29 | Architecture | |
| 12/1 | Microarchitecture | |
| 12/6 | Devices: superconductors | VQE lab part 1 due |
| 12/8 | Devices: ion traps | |
| 12/13 | Conclusion | |

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# Representations for Quantum Computing

## Different representations useful in different settings

1. Quantum circuits
2. Stabilizers
3. Tensor networks
4. Noisy density matrices and Kraus operator sums
5. Logical formulas

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# What makes a quantum circuit difficult to simulate?

## Stabilizers

- Simulation difficulty grows exponentially w.r.t. number of T gates
- A statement about parameters.

## Tensor network contraction

- Simulation difficulty grows exponentially w.r.t. maximum treewidth.
- A statement about topology.

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# Tensors

## Rank-$k$ generalizations of matrices

- Rank-0 tensor: a scalar
- Rank-1 tensor: a vector
- Rank-2 tensor: a matrix
- Rank-3 tensor: ...

# Rank-0 tensor: a scalar

- In quantum circuits, a single amplitude is a complex scalar and therefore a rank-0 tensor.
- For example, a single qubit state is in general $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$. $\alpha$ and $\beta$ are scalars.

# Rank-1 tensor: a vector

- In quantum circuits, a single qubit state is a complex vector and therefore a rank-1 tensor.
- For example, a single qubit state is in general $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, a complex vector.

# Rank-2 tensor: a matrix

## Rank-2 tensors appear as single-qubit gates in quantum circuits

- For example, the Hadamard gate has a unitary matrix of $H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$.
- We can view it as a tensor with two ranks, $m0$ and $m1$ like so:

| m0 | m1 | w |
|---|---|---|
| $\|0\rangle$ | $\|0\rangle$ | $\frac{1}{\sqrt{2}}$ |
| $\|0\rangle$ | $\|1\rangle$ | $\frac{1}{\sqrt{2}}$ |
| $\|1\rangle$ | $\|0\rangle$ | $\frac{1}{\sqrt{2}}$ |
| $\|1\rangle$ | $\|1\rangle$ | $\frac{-1}{\sqrt{2}}$ |

# Rank-2 tensor

## Rank-2 tensors also appear as two-qubit states in quantum circuits

- For example, a two-qubit state is in general

$$|\psi\rangle = \alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}$$

- We can view it as a tensor with two ranks, $q0$ and $q1$ like so:

| q0 | q1 | w |
|---|---|---|
| $\|0\rangle$ | $\|0\rangle$ | $\alpha$ |
| $\|0\rangle$ | $\|1\rangle$ | $\beta$ |
| $\|1\rangle$ | $\|0\rangle$ | $\gamma$ |
| $\|1\rangle$ | $\|1\rangle$ | $\delta$ |

# Rank-4 tensor

## Rank-4 tensors appear as two-qubit gates in quantum circuits

For example, the *CNOT* gate has a unitary matrix of

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

We can view it as a tensor with four ranks, $q0m0$, $q0m1$, $q1m0$, and $q1m1$:

| q0m0 | q0m1 | q1m0 | q1m1 | w |
|---|---|---|---|---|
| $\|0\rangle$ | $\|0\rangle$ | $\|0\rangle$ | $\|0\rangle$ | 1 |
| $\|0\rangle$ | $\|0\rangle$ | $\|0\rangle$ | $\|1\rangle$ | 0 |
| $\|0\rangle$ | $\|0\rangle$ | $\|1\rangle$ | $\|0\rangle$ | 0 |
| $\|0\rangle$ | $\|0\rangle$ | $\|1\rangle$ | $\|1\rangle$ | 1 |
| $\|0\rangle$ | $\|1\rangle$ | $\|0\rangle$ | $\|0\rangle$ | 0 |
| $\|0\rangle$ | $\|1\rangle$ | $\|0\rangle$ | $\|1\rangle$ | 0 |
| $\|0\rangle$ | $\|1\rangle$ | $\|1\rangle$ | $\|0\rangle$ | 0 |
| $\|0\rangle$ | $\|1\rangle$ | $\|1\rangle$ | $\|1\rangle$ | 0 |
| $\|1\rangle$ | $\|0\rangle$ | $\|0\rangle$ | $\|0\rangle$ | 0 |
| $\|1\rangle$ | $\|0\rangle$ | $\|0\rangle$ | $\|1\rangle$ | 0 |
| $\|1\rangle$ | $\|0\rangle$ | $\|1\rangle$ | $\|0\rangle$ | 0 |
| $\|1\rangle$ | $\|0\rangle$ | $\|1\rangle$ | $\|1\rangle$ | 0 |
| $\|1\rangle$ | $\|1\rangle$ | $\|0\rangle$ | $\|0\rangle$ | 0 |
| $\|1\rangle$ | $\|1\rangle$ | $\|0\rangle$ | $\|1\rangle$ | 1 |
| $\|1\rangle$ | $\|1\rangle$ | $\|1\rangle$ | $\|0\rangle$ | 1 |
| $\|1\rangle$ | $\|1\rangle$ | $\|1\rangle$ | $\|1\rangle$ | 0 |

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# Tensor network contraction

- Tensor network contraction is one type of tensor-tensor multiplication.
- It is a generalized form of matrix multiplication.
- Merge two tensors into one. Absorb common edges. If the two tensors share a common index, sum over all possible values of that index.

# Tensor network contraction

For example, we can contract the tensor network for a Bell state circuit

Hadamard gate rank-2 tensor:

| q0m0 | q0m1 | w |
|---|---|---|
| $\|0\rangle$ | $\|0\rangle$ | $\frac{1}{\sqrt{2}}$ |
| $\|0\rangle$ | $\|1\rangle$ | $\frac{1}{\sqrt{2}}$ |
| $\|1\rangle$ | $\|0\rangle$ | $\frac{1}{\sqrt{2}}$ |
| $\|1\rangle$ | $\|1\rangle$ | $\frac{-1}{\sqrt{2}}$ |

CNOT gate rank-4 tensor:

| q0m1 | q0m2 | q1m1 | q1m2 | w |
|---|---|---|---|---|
| $\|0\rangle$ | $\|0\rangle$ | $\|0\rangle$ | $\|0\rangle$ | 1 |
| $\|0\rangle$ | $\|0\rangle$ | $\|0\rangle$ | $\|1\rangle$ | 0 |
| $\|0\rangle$ | $\|0\rangle$ | $\|1\rangle$ | $\|0\rangle$ | 0 |
| $\|0\rangle$ | $\|0\rangle$ | $\|1\rangle$ | $\|1\rangle$ | 1 |
| $\|0\rangle$ | $\|1\rangle$ | $\|0\rangle$ | $\|0\rangle$ | 0 |
| $\|0\rangle$ | $\|1\rangle$ | $\|0\rangle$ | $\|1\rangle$ | 0 |
| $\|0\rangle$ | $\|1\rangle$ | $\|1\rangle$ | $\|0\rangle$ | 0 |
| $\|0\rangle$ | $\|1\rangle$ | $\|1\rangle$ | $\|1\rangle$ | 0 |
| $\|1\rangle$ | $\|0\rangle$ | $\|0\rangle$ | $\|0\rangle$ | 0 |
| $\|1\rangle$ | $\|0\rangle$ | $\|0\rangle$ | $\|1\rangle$ | 0 |
| $\|1\rangle$ | $\|0\rangle$ | $\|1\rangle$ | $\|0\rangle$ | 0 |
| $\|1\rangle$ | $\|0\rangle$ | $\|1\rangle$ | $\|1\rangle$ | 0 |
| $\|1\rangle$ | $\|1\rangle$ | $\|0\rangle$ | $\|0\rangle$ | 0 |
| $\|1\rangle$ | $\|1\rangle$ | $\|0\rangle$ | $\|1\rangle$ | 1 |
| $\|1\rangle$ | $\|1\rangle$ | $\|1\rangle$ | $\|0\rangle$ | 1 |
| $\|1\rangle$ | $\|1\rangle$ | $\|1\rangle$ | $\|1\rangle$ | 0 |

# Tensor network contraction

Contract tensors by summing over $q0m1$:

| q0m0 | q0m2 | q1m1 | q1m2 | w |
|---|---|---|---|---|
| $\|0\rangle$ | $\|0\rangle$ | $\|0\rangle$ | $\|0\rangle$ | $\frac{1}{\sqrt{2}}$ |
| $\|0\rangle$ | $\|0\rangle$ | $\|0\rangle$ | $\|1\rangle$ | 0 |
| $\|0\rangle$ | $\|0\rangle$ | $\|1\rangle$ | $\|0\rangle$ | 0 |
| $\|0\rangle$ | $\|0\rangle$ | $\|1\rangle$ | $\|1\rangle$ | $\frac{1}{\sqrt{2}}$ |
| $\|0\rangle$ | $\|1\rangle$ | $\|0\rangle$ | $\|0\rangle$ | 0 |
| $\|0\rangle$ | $\|1\rangle$ | $\|0\rangle$ | $\|1\rangle$ | $\frac{1}{\sqrt{2}}$ |
| $\|0\rangle$ | $\|1\rangle$ | $\|1\rangle$ | $\|0\rangle$ | $\frac{1}{\sqrt{2}}$ |
| $\|0\rangle$ | $\|1\rangle$ | $\|1\rangle$ | $\|1\rangle$ | 0 |
| $\|1\rangle$ | $\|0\rangle$ | $\|0\rangle$ | $\|0\rangle$ | $\frac{1}{\sqrt{2}}$ |
| $\|1\rangle$ | $\|0\rangle$ | $\|0\rangle$ | $\|1\rangle$ | 0 |
| $\|1\rangle$ | $\|0\rangle$ | $\|1\rangle$ | $\|0\rangle$ | 0 |
| $\|1\rangle$ | $\|0\rangle$ | $\|1\rangle$ | $\|1\rangle$ | $\frac{1}{\sqrt{2}}$ |
| $\|1\rangle$ | $\|1\rangle$ | $\|0\rangle$ | $\|0\rangle$ | 0 |
| $\|1\rangle$ | $\|1\rangle$ | $\|0\rangle$ | $\|1\rangle$ | $\frac{-1}{\sqrt{2}}$ |
| $\|1\rangle$ | $\|1\rangle$ | $\|1\rangle$ | $\|0\rangle$ | $\frac{-1}{\sqrt{2}}$ |
| $\|1\rangle$ | $\|1\rangle$ | $\|1\rangle$ | $\|1\rangle$ | 0 |

Compare this with unitary matrix:

$$CNOT(H \otimes I)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$

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# Tensor network contraction order

Contraction ordering says the order in which edges are contracted.

- To minimize computation and memory requirements, best to avoid forming large intermediate tensors.
- Akin to the classic dynamic programming problem of optimal chain matrix multiplication.

# Tensor network contraction order

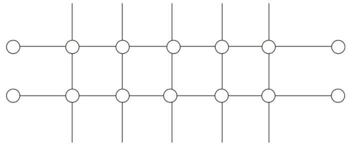

Figure 9.4: Part of a generic tensor network, consisting of ten rank-4 tensors and four rank-1 tensors.

Figure: Source: [Ding and Chong, 2020]

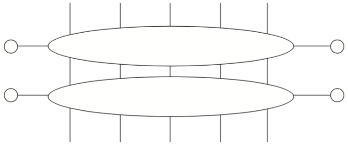

Figure 9.5: First strategy of contraction that results in two rank-12 tensors and four rank-1 tensors. Then contracting the two rank-12 tensors involves contracting 5 edges at once, by summing over $2^5$ terms.

Figure: Source: [Ding and Chong, 2020]

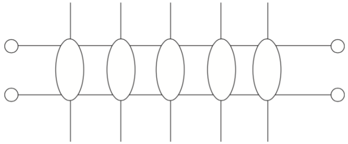

Figure 9.6: Second strategy of contraction that results in five rank-6 tensors and four rank-1 tensors. Then contracting the five rank-6 tensors involves contracting from left to right 2 edges at a time, by summing over $2^2$ terms four times.

Figure: Source: [Ding and Chong, 2020]

# Tensor network contraction order

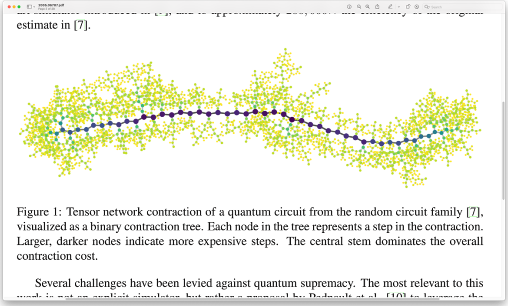
estimate in [7].

Figure 1: Tensor network contraction of a quantum circuit from the random circuit family [7], visualized as a binary contraction tree. Each node in the tree represents a step in the contraction. Larger, darker nodes indicate more expensive steps. The central stem dominates the overall contraction cost.

Several challenges have been levied against quantum supremacy. The most relevant to this

Figure: Source:Cupjin Huang et al., Classical Simulation of Quantum Supremacy Circuits, 2020. [Huang et al., 2020]

Cost of simulating the quantum circuit is via tensor network contraction is O(exp(treewidth)) [Markov and Shi, 2008]

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# Unification of stabilizers and tensors

- If you feed a Clifford circuit to a tensor network contraction based simulator, it will not see Clifford symmetry
- Need some way to enable Clifford simplification of tensor networks.

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# Example: inverting a CNOT

## What is this circuit: $(H \otimes H)CNOT_{0,1}(H \otimes H)$?

$$
\begin{aligned}
&(H \otimes H)CNOT_{0,1}(H \otimes H) \\
&= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
&= CNOT_{1,0}
\end{aligned}
$$

All gates here ($H$, $CNOT$) in Clifford gate set. Automatic simplification method?

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# Splitting a CNOT into network of two rank-3 tensors

"Copy" rank-3 tensor

| a | b | c | w |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

"XOR" rank-3 tensor

| c | d | e | w |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 0 |

Contract "Copy" with "XOR":

- Sum over $c$.
- Gives the *CNOT* rank-4 tensor.

| a | b | d | e | w |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 0 |

[Biamonte and Bergholm, 2017], [Biamonte, 2019]

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# Tensor simplification rules: Duality of Copy and XOR tensors.

Hadamard rank-2 tensor:

| a | b | w |
|---|---|---|
| 0 | 0 | $\frac{1}{\sqrt{2}}$ |
| 0 | 1 | $\frac{1}{\sqrt{2}}$ |
| 1 | 0 | $\frac{1}{\sqrt{2}}$ |
| 1 | 1 | $\frac{-1}{\sqrt{2}}$ |

"Copy" rank-3 tensor:

| b | c | e | w |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

Contraction of $H$ with Copy summing over b:

| a | c | e | w |
|---|---|---|---|
| 0 | 0 | 0 | $\frac{1}{\sqrt{2}}$ |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | $\frac{1}{\sqrt{2}}$ |
| 1 | 0 | 0 | $\frac{1}{\sqrt{2}}$ |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | $\frac{-1}{\sqrt{2}}$ |

# Tensor simplification rules: Duality of Copy and XOR tensors.

Hadamard rank-2 tensor:

| c | d | w |
|---|---|---|
| 0 | 0 | $\frac{1}{\sqrt{2}}$ |
| 0 | 1 | $\frac{1}{\sqrt{2}}$ |
| 1 | 0 | $\frac{1}{\sqrt{2}}$ |
| 1 | 1 | $\frac{-1}{\sqrt{2}}$ |

Contraction of $H$ with Copy summing over b:

| a | c | e | w |
|---|---|---|---|
| 0 | 0 | 0 | $\frac{1}{\sqrt{2}}$ |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | $\frac{1}{\sqrt{2}}$ |
| 1 | 0 | 0 | $\frac{1}{\sqrt{2}}$ |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | $\frac{-1}{\sqrt{2}}$ |

Contraction of $H$ with {contraction of $H$ with Copy summing over b} summing over c:

| a | d | e | w |
|---|---|---|---|
| 0 | 0 | 0 | $\frac{1}{2}$ |
| 0 | 0 | 1 | $\frac{1}{2}$ |
| 0 | 1 | 0 | $\frac{1}{2}$ |
| 0 | 1 | 1 | $\frac{-1}{2}$ |
| 1 | 0 | 0 | $\frac{1}{2}$ |
| 1 | 0 | 1 | $\frac{-1}{2}$ |
| 1 | 1 | 0 | $\frac{1}{2}$ |
| 1 | 1 | 1 | $\frac{1}{2}$ |

# Tensor simplification rules: Duality of Copy and XOR tensors.

Hadamard rank-2 tensor:

| e | f | w |
|---|---|---|
| 0 | 0 | $\frac{1}{\sqrt{2}}$ |
| 0 | 1 | $\frac{1}{\sqrt{2}}$ |
| 1 | 0 | $\frac{1}{\sqrt{2}}$ |
| 1 | 1 | $\frac{-1}{\sqrt{2}}$ |

Contraction of $H$ with {contraction of $H$ with Copy summing over b} summing over c:

| a | d | e | w |
|---|---|---|---|
| 0 | 0 | 0 | $\frac{1}{2}$ |
| 0 | 0 | 1 | $\frac{1}{2}$ |
| 0 | 1 | 0 | $\frac{1}{2}$ |
| 0 | 1 | 1 | $\frac{-1}{2}$ |
| 1 | 0 | 0 | $\frac{1}{2}$ |
| 1 | 0 | 1 | $\frac{-1}{2}$ |
| 1 | 1 | 0 | $\frac{1}{2}$ |
| 1 | 1 | 1 | $\frac{1}{2}$ |

Contraction of $H$ with {contraction of $H$ with {contraction of $H$ with Copy summing over b} summing over c} summing over e:

| a | d | f | w |
|---|---|---|---|
| 0 | 0 | 0 | $\frac{1}{\sqrt{2}}$ |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | $\frac{1}{\sqrt{2}}$ |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | $\frac{1}{\sqrt{2}}$ |
| 1 | 1 | 0 | $\frac{1}{\sqrt{2}}$ |
| 1 | 1 | 1 | 0 |

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# Automatic simplification of circuits

What is this circuit: $(H \otimes H)CNOT_{0,1}(H \otimes H)$?

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