Quantum computing fundamentals: single qubits, multiple qubits, dynamics

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Table of contents

Postulates of quantum mechanics

The state of a single qubit
- Bloch sphere

The state of multiple qubits
- Tensor product

The evolution of qubit states

Postulates of quantum mechanics

1. State space
2. Composite systems
3. Evolution
4. Quantum measurement

Quantum postulate 1: State space

The state of an isolated physical system is described as a wavefunction

- Assuming continuous state space:

$$|\psi\rangle = \int_{-\infty}^{\infty} \psi(x)\,|x\rangle\,dx$$

$$\psi(x) \in \mathbb{C}$$

- Assuming discrete state space:

$$|\psi\rangle = \sum_{i=0}^{\infty} \psi(x_i)\,|x_i\rangle$$

$$\psi(x) \in \mathbb{C}$$

Quantum postulate 1: State space

The position or momentum of a physical system is described as a wavefunction

- Assuming discrete state space:

$$|\psi\rangle = \sum_{i=0}^{\infty} \psi(x_i)\,|x_i\rangle$$

$$\psi(x) \in \mathbb{C}$$

- Assuming discrete binarized state space:

$$|\psi\rangle = \sum_{i=0}^{1} \psi(x_i)\,|x_i\rangle$$

$$\psi(x) \in \mathbb{C}$$

Table of contents

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The state of a single qubit

Single qubit state

- $\alpha |0\rangle + \beta |1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$
- Amplitudes $\alpha, \beta \in \mathbb{C}$
- $|\alpha|^2 + |\beta|^2 = 1$

Many physical phenomena can be in superposition and encode qubits

- Polarization of light in different directions
- Electron spins (Intel solid state qubits)
- Atom energy states (UMD, IonQ ion trap qubits)
- Quantized voltage and current (IBM, Google superconducting qubits)

If multiple discrete values are possible (e.g., atom energy states, voltage and current), we pick (bottom) two for the binary abstraction.

Bloch sphere

Representation of pure states of a single qubit

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

- θ polar angle
- ϕ azimuthal angle

Euler's formula

$$e^{i\phi} = \cos\phi + i\sin\phi$$

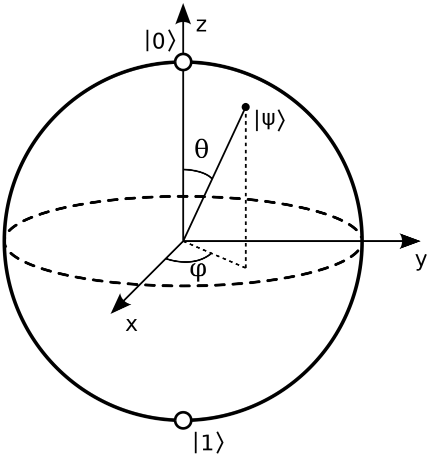


Figure: Source: Wikimedia

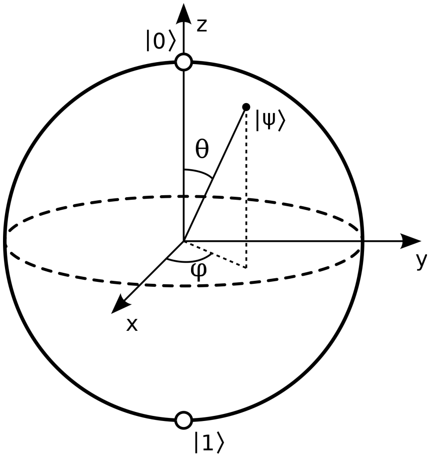
Bloch sphere

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

$$e^{i\phi} = \cos\phi + i\sin\phi$$

Important locations on the Bloch sphere

- $|+\rangle = H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
- $|-\rangle = H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$



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Bloch sphere

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

$$e^{i\phi} = \cos\phi + i\sin\phi$$

Rotations around the Bloch sphere

- $$R_x(\theta) = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}X$$
- $$R_y(\theta) = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Y$$
- $$R_x(\theta) = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Z$$

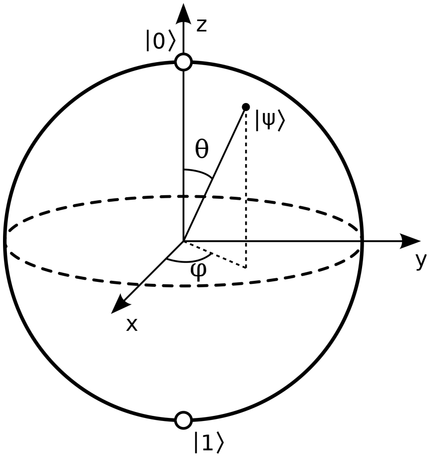

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Table of contents

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Quantum postulate 2: Composite systems

The state space of composite systems is the tensor product of state space of component systems.

Multiple qubits: the tensor product

Tensor product (also known as Kronecker product) of state vectors

$$|+\rangle \otimes |-\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \\ \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2} \\ \frac{1}{2} \\ \frac{-1}{2} \end{bmatrix} = \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

Table of contents

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Quantum postulate 3: Evolution

The time evolution of a state follows the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle$$

- Comes from the conservation of total energy in the closed system, one of the observables from the system state.
- Itself reflects a time-invariance.

$$\frac{\partial}{\partial t} |\psi(t)\rangle = \frac{-iH}{\hbar} |\psi(t)\rangle$$

$$|\psi(t)\rangle = e^{\frac{-iH}{\hbar}} |\psi(t)\rangle$$

H must be normal and have only real eigenvalues. Therefore, H is Hermitian (self-adjoint), $H^\dagger = H$