Quantum computing fundamentals: dynamics, measurement

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Postulates of quantum mechanics

1. State space
2. Composite systems
3. Evolution
4. Quantum measurement

1, 2, and 3 are linear and describe closed quantum systems. 4 is nonlinear and describes open quantum systems.

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Quantum postulate 3: Evolution

The time evolution of a state follows the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

- Comes from the conservation of total energy in the closed system, one of the observables from the system state.
- Itself reflects a time-invariance.

$$\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{-iH}{\hbar}|\psi(t)\rangle$$

$$|\psi(t)\rangle = e^{\frac{-iH}{\hbar}}|\psi(t)\rangle$$

Quantum postulate 3: Evolution

The evolution of a closed quantum system is a unitary transformation.

$$|\psi(t = t_1)\rangle = U\,|\psi(t = t_0)\rangle$$

- $|\psi_1\rangle = U\,|\psi_0\rangle$
- In a closed quantum system, $\langle\psi_1|\,|\psi_1\rangle = \langle\psi_0|\,U^\dagger U\,|\psi_0\rangle = \langle\psi_0|\,|\psi_0\rangle = 1$
- $U^\dagger U = I$, $U^\dagger = U^{-1}$; Such matrices U are unitary

Quantum postulate 3: Evolution

From unitary transformations we can show Hamiltonians in closed quantum systems must be hermitian

- $U\left|\psi\right\rangle = e^{\frac{-iH}{\hbar}}\left|\psi\right\rangle$
- $U^{\dagger}\left|\psi\right\rangle = e^{\frac{-(iH)^{\dagger}}{\hbar}}\left|\psi\right\rangle$
- $U^{\dagger}\left|\psi\right\rangle = U^{-1}\left|\psi\right\rangle = e^{\frac{iH}{\hbar}}\left|\psi\right\rangle$
- $(iH)^{\dagger} = -iH$, $A = iH$; such matrices A are called anti-Hermitian a.k.a. skew-Hermitian
- If iH is skew-Hermitian, H is Hermitian a.k.a. self-adjoint: $H^{\dagger} = H$

No-cloning theorem

There is no way to duplicate an arbitrary quantum state

Suppose a cloning operation U_c exists. Then:

- $$U_c(|\phi\rangle \otimes |\omega\rangle) = |\phi\rangle \otimes |\phi\rangle\,,$$
  $$U_c(|\psi\rangle \otimes |\omega\rangle) = |\psi\rangle \otimes |\psi\rangle\,,$$
  for arbitrary states $|\phi\rangle\,, |\psi\rangle$ we wish to copy.
- The overlap of the initial states is:

$$\langle\phi| \otimes \langle\omega|\, |\psi\rangle \otimes |\omega\rangle = \langle\phi|\,|\psi\rangle \cdot \langle\omega|\,|\omega\rangle = \langle\phi|\,|\psi\rangle$$

No-cloning theorem

There is no way to duplicate an arbitrary quantum state

Suppose a cloning operation U_c exists. Then:

- $$U_c(|\phi\rangle \otimes |\omega\rangle) = |\phi\rangle \otimes |\phi\rangle\,,$$
  $$U_c(|\psi\rangle \otimes |\omega\rangle) = |\psi\rangle \otimes |\psi\rangle\,,$$

  for arbitrary states $|\phi\rangle\,, |\psi\rangle$ we wish to copy.
- The overlap of the final states is:

  $$\langle\phi| \otimes \langle\phi|\,|\psi\rangle \otimes |\psi\rangle = \langle\phi|\,|\psi\rangle \cdot \langle\phi|\,|\psi\rangle = (\langle\phi|\,|\psi\rangle)^2$$

- The overlap of the final states is also:

  $$\langle\phi| \otimes \langle\phi|\,|\psi\rangle \otimes |\psi\rangle = \langle\phi| \otimes \langle\omega|\,U^\dagger U\,|\psi\rangle \otimes |\omega\rangle = \langle\phi| \otimes \langle\omega|\,|\psi\rangle \otimes |\omega\rangle = \langle\phi|\,|\psi\rangle$$

- $(\langle\phi|\,|\psi\rangle)^2 = \langle\phi|\,|\psi\rangle$, so $\langle\phi|\,|\psi\rangle = 0$, or $\langle\phi|\,|\psi\rangle = 1$, $|\phi\rangle$ and $|\psi\rangle$ cannot be arbitrary states as claimed.

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Quantum postulate 3: Measurement

When a closed quantum system with state $|\psi\rangle$ interacts with the environment, measurement takes place:

- The probability of the post-measurement state being in state $|a_n\rangle$ is: $p(|a_n\rangle) = \langle\psi| |a_n\rangle \langle a_n| |\psi\rangle = |\langle a_n| |\psi\rangle|^2$
- The state of the quantum system is then renormalized to $\frac{|a_n\rangle\langle a_n||\psi\rangle}{\sqrt{p(|a_n\rangle)}}$

Let's practice this with measuring a single-qubit state.

Entangled states: Bell state circuit

Bell state circuit

$$|00\rangle \xrightarrow{H \otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |\Phi^+\rangle$$

Can $|\Phi^+\rangle$ be treated as the tensor product (composition) of two individual qubits?

Prove that the Bell state cannot be factored into two single-qubit states

Bell state circuit

$$|00\rangle \xrightarrow{H \otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = |\Phi^+\rangle$$

Can $|\Phi^+\rangle$ be treated as the tensor product (composition) of two individual qubits?

No.

Bell states form an orthogonal basis set

1. $|00\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) = |\Phi^+\rangle$
2. $|01\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|01\rangle + |11\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|01\rangle + |10\rangle \Big) = |\Psi^+\rangle$
3. $|10\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle - |10\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle - |11\rangle \Big) = |\Phi^-\rangle$
4. $|11\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|01\rangle - |11\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|01\rangle - |10\rangle \Big) = |\Psi^-\rangle$