# Basic quantum algorithms: teleportation, Bell's inequality

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# Bell states form an orthogonal basis set

1. $|00\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) = |\Phi^+\rangle$
2. $|01\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|01\rangle + |11\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|01\rangle + |10\rangle\Big) = |\Psi^+\rangle$
3. $|10\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle - |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle - |11\rangle\Big) = |\Phi^-\rangle$
4. $|11\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|01\rangle - |11\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|01\rangle - |10\rangle\Big) = |\Psi^-\rangle$

# Superdense coding

## Transmit 2 bits of classical information by sending 1 qubit

1. Alice wishes to tell Bob two bits of information: 00, 01, 10, or 11.
2. Alice and Bob each have one qubit of a Bell pair in state $|P\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$.
3. Alice performs $I$, $X$, $Z$, or $ZX$ on her qubit; she then sends her qubit to Bob.
4. Bob measures in the Bell basis to receive 00, 01, 10, or 11.

## Superdense coding circuit

`https://github.com/quantumlib/Cirq/blob/master/examples/superdense_coding.py`

# Superdense coding

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4. Bob measures in the Bell basis to receive 00, 01, 10, or 11.

## Alice applies different operators on her qubit so Bob measures the message

1. $|P\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) \xrightarrow{I\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle\Big) \xrightarrow{H\otimes I} |00\rangle$
2. $|P\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) \xrightarrow{X\otimes I} \frac{1}{\sqrt{2}}\Big(|10\rangle + |01\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|11\rangle + |01\rangle\Big) \xrightarrow{H\otimes I} |01\rangle$
3. $|P\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) \xrightarrow{Z\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle - |11\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle - |10\rangle\Big) \xrightarrow{H\otimes I} |10\rangle$
4. $|P\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) \xrightarrow{ZX\otimes I} \frac{1}{\sqrt{2}}\Big(-|10\rangle + |01\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(-|11\rangle + |01\rangle\Big) \xrightarrow{H\otimes I} |11\rangle$

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# Quantum teleportation

## "Teleport" a qubit state by transmitting classical information

1. Alice wishes to give Bob a qubit state $|Q\rangle$.
2. Alice and Bob each have one qubit of a Bell pair in state $|P\rangle$.
3. Alice first entangles $|Q\rangle$ and $|P\rangle$; then, she measures her local two qubits.
4. Alice tells Bob (via classical means) her two-bit measurement result.
5. Bob uses Alice's two bits to perform $I$, $X$, $Z$, or $ZX$ on his qubit to obtain $|Q\rangle$.

## Step-by-step qubit state calculation up to Alice's measurement

$$
\begin{aligned}
|Q\rangle \otimes |P\rangle &= \big(\alpha\,|0\rangle + \beta\,|1\rangle\big) \otimes \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big) \\
&= \frac{\alpha}{\sqrt{2}}\,|0\rangle\,\big(|00\rangle + |11\rangle\big) + \frac{\beta}{\sqrt{2}}\,|1\rangle\,\big(|00\rangle + |11\rangle\big) \\
&\xrightarrow{CNOT_{0,1}} \frac{\alpha}{\sqrt{2}}\,|0\rangle\,\big(|00\rangle + |11\rangle\big) + \frac{\beta}{\sqrt{2}}\,|1\rangle\,\big(|10\rangle + |01\rangle\big) \\
&\xrightarrow{H\otimes I\otimes I} \frac{\alpha}{2}\big(|0\rangle + |1\rangle\big)\big(|00\rangle + |11\rangle\big) + \frac{\beta}{2}\big(|0\rangle - |1\rangle\big)\big(|10\rangle + |01\rangle\big) \\
&= \frac{1}{2}\,|00\rangle\,\big(\alpha\,|0\rangle + \beta\,|1\rangle\big) \\
&+ \frac{1}{2}\,|01\rangle\,\big(\alpha\,|1\rangle + \beta\,|0\rangle\big) \\
&+ \frac{1}{2}\,|10\rangle\,\big(\alpha\,|0\rangle - \beta\,|1\rangle\big) \\
&+ \frac{1}{2}\,|11\rangle\,\big(\alpha\,|1\rangle - \beta\,|0\rangle\big)
\end{aligned}
$$

# Quantum teleportation

## "Teleport" a qubit state by transmitting classical information

1. Alice wishes to give Bob a qubit state $|Q\rangle$.
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3. Alice first entangles $|Q\rangle$ and $|P\rangle$; then, she measures her local two qubits.
4. Alice tells Bob (via classical means) her two-bit measurement result.
5. Bob uses Alice's two bits to perform $I$, $X$, $Z$, or $ZX$ on his qubit to obtain $|Q\rangle$.

## Depending on if Alice measures 00, 01, 10, or 11, Bob applies I, X, Z, or ZX to recover $|Q\rangle$

$$
\begin{aligned}
|Q\rangle \otimes |P\rangle &= \left(\alpha\,|0\rangle + \beta\,|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \\
&= \frac{\alpha}{\sqrt{2}}\,|0\rangle\left(|00\rangle + |11\rangle\right) + \frac{\beta}{\sqrt{2}}\,|1\rangle\left(|00\rangle + |11\rangle\right) \\
&\xrightarrow{CNOT_{0,1}} \frac{\alpha}{\sqrt{2}}\,|0\rangle\left(|00\rangle + |11\rangle\right) + \frac{\beta}{\sqrt{2}}\,|1\rangle\left(|10\rangle + |01\rangle\right) \\
&\xrightarrow{H\otimes I\otimes I} \frac{\alpha}{2}\left(|0\rangle + |1\rangle\right)\left(|00\rangle + |11\rangle\right) + \frac{\beta}{2}\left(|0\rangle - |1\rangle\right)\left(|10\rangle + |01\rangle\right) \\
&= \frac{1}{2}\,|00\rangle\left(\alpha\,|0\rangle + \beta\,|1\rangle\right) \quad \text{Alice measures 00 so Bob applies I} \\
&+ \frac{1}{2}\,|01\rangle\left(\alpha\,|1\rangle + \beta\,|0\rangle\right) \quad \text{Alice measures 01 so Bob applies X} \\
&+ \frac{1}{2}\,|10\rangle\left(\alpha\,|0\rangle - \beta\,|1\rangle\right) \quad \text{Alice measures 10 so Bob applies Z} \\
&+ \frac{1}{2}\,|11\rangle\left(\alpha\,|1\rangle - \beta\,|0\rangle\right) \quad \text{Alice measures 11 so Bob applies ZX}
\end{aligned}
$$

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# Test of entanglement

## Two isolated parties Alice and Bob

- Alice gets coin toss x, replies a
- Bob gets coin toss y, replies b

Goal: maximize $a \oplus b = x \wedge y$

| $x$ | $y$ | $x \wedge y$ | $a \oplus b$ | winning options for $(a, b)$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | (0,0) or (1,1) |
| 0 | 1 | 0 | 0 | (0,0) or (1,1) |
| 1 | 0 | 0 | 0 | (0,0) or (1,1) |
| 1 | 1 | 1 | 1 | (0,1) or (1,0) |

# Best classical strategy to maximize $a \oplus b = x \wedge y$

## Proof that any assignment to $a$ and $b$ cannot always satisfy $a \oplus b = x \wedge y$

1. Let $a_0$ be Alice's response if she sees $x = 0$
2. Let $a_1$ be Alice's response if she sees $x = 1$
3. Let $b_0$ be Bob's response if she sees $y = 0$
4. Let $b_1$ be Bob's response if she sees $y = 1$

## Satisfy $a \oplus b = x \wedge y$

1. $a_0 \oplus b_0 = 0$
2. $a_0 \oplus b_1 = 0$
3. $a_1 \oplus b_0 = 0$
4. $a_1 \oplus b_1 = 1$

## Sum (mod 2) of left side

$$(a_0 \oplus b_0) \oplus (a_0 \oplus b_1) \oplus (a_1 \oplus b_0) \oplus (a_1 \oplus b_1) =$$
$$(a_0 \oplus a_0) \oplus (a_1 \oplus a_1) \oplus (b_0 \oplus b_0) \oplus (b_1 \oplus b_1) = 0$$

## Sum (mod 2) of right side

1

# Best classical strategy to maximize $a \oplus b = x \wedge y$

Even if the two shared randomness, the random coin toss of x and y prevents use of shared randomness.
Best you can do is 3/4. Give a couple ways of getting 3/4

# A quantum strategy to maximize $a \oplus b = x \wedge y$

Alice and Bob share entangled pair $|\Phi\rangle$

$|\Phi\rangle = \frac{1}{\sqrt{12}}\Big(3\,|00\rangle + |01\rangle + |10\rangle - |11\rangle\Big)$

$(x, y) = (0, 0)$ So Alice and Bob both apply $I$:

$$(I \otimes I)\,|\Phi\rangle = \frac{1}{\sqrt{12}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Measurement yields

$$\begin{cases} (a, b) = (0, 0), \text{ a win, with probability } \frac{9}{12} \\ (a, b) = (0, 1), \text{ a loss, with probability } \frac{1}{12} \\ (a, b) = (1, 0), \text{ a loss, with probability } \frac{1}{12} \\ (a, b) = (1, 1), \text{ a win, with probability } \frac{1}{12} \end{cases}$$

# A quantum strategy to maximize $a \oplus b = x \wedge y$

Alice and Bob share entangled pair $|\Phi\rangle$

$|\Phi\rangle = \frac{1}{\sqrt{12}}\Big(3\,|00\rangle + |01\rangle + |10\rangle - |11\rangle\Big)$

$(x, y) = (0, 1)$ So Alice applies $I$, Bob applies $H$:

$$(I \otimes H)\,|\Phi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\end{bmatrix}\frac{1}{\sqrt{12}}\begin{bmatrix}3\\ 1\\ 1\\ -1\end{bmatrix} = \frac{1}{2\sqrt{6}}\begin{bmatrix}4\\ 2\\ 0\\ 2\end{bmatrix}$$

Measurement yields

$$\begin{cases}(a, b) = (0, 0), \text{ a win, with probability } \frac{4}{6}\\ (a, b) = (0, 1), \text{ a loss, with probability } \frac{1}{6}\\ (a, b) = (1, 0), \text{ a loss, with probability } 0\\ (a, b) = (1, 1), \text{ a win, with probability } \frac{1}{6}\end{cases}$$

# A quantum strategy to maximize $a \oplus b = x \wedge y$

Alice and Bob share entangled pair $|\Phi\rangle$

$|\Phi\rangle = \frac{1}{\sqrt{12}}\Big(3|00\rangle + |01\rangle + |10\rangle - |11\rangle\Big)$

$(x, y) = (1, 0)$ So Alice applies $H$, Bob applies $I$:

$$(H \otimes I)|\Phi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \frac{1}{\sqrt{12}}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2\sqrt{6}}\begin{bmatrix} 4 \\ 0 \\ 2 \\ 2 \end{bmatrix}$$

Measurement yields

$$\begin{cases} (a, b) = (0, 0), \text{ a win, with probability } \frac{4}{6} \\ (a, b) = (0, 1), \text{ a loss, with probability } 0 \\ (a, b) = (1, 0), \text{ a loss, with probability } \frac{1}{6} \\ (a, b) = (1, 1), \text{ a win, with probability } \frac{1}{6} \end{cases}$$

# A quantum strategy to maximize $a \oplus b = x \wedge y$

Alice and Bob share entangled pair $|\Phi\rangle$

$|\Phi\rangle = \frac{1}{\sqrt{12}}\Big(3\,|00\rangle + |01\rangle + |10\rangle - |11\rangle\Big)$

$(x, y) = (1, 1)$ So Alice and Bob both apply $H$:

$$(H \otimes H)\,|\Phi\rangle = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \frac{1}{\sqrt{12}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{4\sqrt{3}} \begin{bmatrix} 4 \\ 4 \\ 4 \\ 0 \end{bmatrix}$$

Measurement yields

$$\begin{cases} (a, b) = (0, 0), \text{ a loss, with probability } \frac{1}{3} \\ (a, b) = (0, 1), \text{ a win, with probability } \frac{1}{3} \\ (a, b) = (1, 0), \text{ a win, with probability } \frac{1}{3} \\ (a, b) = (1, 1), \text{ a loss, with probability } 0 \end{cases}$$