Basic quantum algorithms: teleportation, Bell’s inequality

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Quantum cryptography / quantum key exchange / BB84

Entanglement protocol: Quantum teleportation

Bell inequality testing protocol / game
Bell states form an orthogonal basis set

1. $|00\rangle \xrightarrow{H \otimes I} \frac{1}{\sqrt{2}} \left( |00\rangle + |10\rangle \right) \xrightarrow{CNOT} \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right) = |\Phi^+\rangle$

2. $|01\rangle \xrightarrow{H \otimes I} \frac{1}{\sqrt{2}} \left( |01\rangle + |11\rangle \right) \xrightarrow{CNOT} \frac{1}{\sqrt{2}} \left( |01\rangle + |10\rangle \right) = |\Psi^+\rangle$

3. $|10\rangle \xrightarrow{H \otimes I} \frac{1}{\sqrt{2}} \left( |00\rangle - |10\rangle \right) \xrightarrow{CNOT} \frac{1}{\sqrt{2}} \left( |00\rangle - |11\rangle \right) = |\Phi^-\rangle$

4. $|11\rangle \xrightarrow{H \otimes I} \frac{1}{\sqrt{2}} \left( |01\rangle - |11\rangle \right) \xrightarrow{CNOT} \frac{1}{\sqrt{2}} \left( |01\rangle - |10\rangle \right) = |\Psi^-\rangle$
Superdense coding

Transmit 2 bits of classical information by sending 1 qubit

1. Alice wishes to tell Bob two bits of information: 00, 01, 10, or 11.
2. Alice and Bob each have one qubit of a Bell pair in state 
   \[ |P\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right). \]
3. Alice performs \( I, X, Z, \) or \( ZX \) on her qubit; she then sends her qubit to Bob.
4. Bob measures in the Bell basis to receive 00, 01, 10, or 11.

Superdense coding circuit

https://github.com/quantumlib/Cirq/blob/master/examples/superdense_coding.py
Superdense coding

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3. Alice performs \( I, X, Z, \text{ or } ZX \) on her qubit; she then sends her qubit to Bob.
4. Bob measures in the Bell basis to receive 00, 01, 10, or 11.

Alice applies different operators on her qubit so Bob measures the message

1. \[ |P\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right) \xrightarrow{I \otimes I} \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}} \left( |00\rangle + |10\rangle \right) \xrightarrow{H \otimes I} |00\rangle \]
2. \[ |P\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right) \xrightarrow{X \otimes I} \frac{1}{\sqrt{2}} \left( |10\rangle + |01\rangle \right) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}} \left( |11\rangle + |01\rangle \right) \xrightarrow{H \otimes I} |01\rangle \]
3. \[ |P\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right) \xrightarrow{Z \otimes I} \frac{1}{\sqrt{2}} \left( |00\rangle - |11\rangle \right) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}} \left( |00\rangle - |10\rangle \right) \xrightarrow{H \otimes I} |10\rangle \]
4. \[ |P\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right) \xrightarrow{ZX \otimes I} \frac{1}{\sqrt{2}} \left( |00\rangle - |10\rangle + |01\rangle \right) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}} \left( |11\rangle + |01\rangle \right) \xrightarrow{H \otimes I} |11\rangle \]
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Quantum teleportation

“Teleport” a qubit state by transmitting classical information

1. Alice wishes to give Bob a qubit state \(|Q\rangle\).
2. Alice and Bob each have one qubit of a Bell pair in state \(|P\rangle\).
3. Alice first entangles \(|Q\rangle\) and \(|P\rangle\); then, she measures her local two qubits.
4. Alice tells Bob (via classical means) her two-bit measurement result.
5. Bob uses Alice’s two bits to perform \(I, X, Z, \) or \(ZX\) on his qubit to obtain \(|Q\rangle\).

Step-by-step qubit state calculation up to Alice’s measurement

\[
|Q\rangle \otimes |P\rangle = \left( \alpha |0\rangle + \beta |1\rangle \right) \otimes \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right)
\]

\[
= \frac{\alpha}{\sqrt{2}} |0\rangle \left( |00\rangle + |11\rangle \right) + \frac{\beta}{\sqrt{2}} |1\rangle \left( |00\rangle + |11\rangle \right)
\]

\[
\xrightarrow{CNOT_{0,1}} \frac{\alpha}{\sqrt{2}} |0\rangle \left( |00\rangle + |11\rangle \right) + \frac{\beta}{\sqrt{2}} |1\rangle \left( |10\rangle + |01\rangle \right)
\]

\[
\xrightarrow{H \otimes I \otimes I} \frac{\alpha}{2} \left( |0\rangle + |1\rangle \right) \left( |00\rangle + |11\rangle \right) + \frac{\beta}{2} \left( |0\rangle - |1\rangle \right) \left( |10\rangle + |01\rangle \right)
\]

\[
= \frac{1}{2} |00\rangle \left( \alpha |0\rangle + \beta |1\rangle \right) + \frac{1}{2} |01\rangle \left( \alpha |1\rangle + \beta |0\rangle \right) + \frac{1}{2} |10\rangle \left( \alpha |0\rangle - \beta |1\rangle \right) + \frac{1}{2} |11\rangle \left( \alpha |1\rangle - \beta |0\rangle \right)
\]
Quantum teleportation

“Teleport” a qubit state by transmitting classical information

1. Alice wishes to give Bob a qubit state $|Q\rangle$.
2. Alice and Bob each have one qubit of a Bell pair in state $|P\rangle$.
3. Alice first entangles $|Q\rangle$ and $|P\rangle$; then, she measures her local two qubits.
4. Alice tells Bob (via classical means) her two-bit measurement result.
5. Bob uses Alice’s two bits to perform $I$, $X$, $Z$, or $ZX$ on his qubit to obtain $|Q\rangle$.

Depending on if Alice measures $00$, $01$, $10$, or $11$, Bob applies $I$, $X$, $Z$, or $ZX$ to recover $|Q\rangle$.

\[
|Q\rangle \otimes |P\rangle = \left( \alpha |0\rangle + \beta |1\rangle \right) \otimes \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right)
\]

\[
= \frac{\alpha}{\sqrt{2}} |0\rangle \left( |00\rangle + |11\rangle \right) + \frac{\beta}{\sqrt{2}} |1\rangle \left( |00\rangle + |11\rangle \right)
\]

\[
\xrightarrow{\text{CNOT}_{0,1}} \frac{\alpha}{\sqrt{2}} |0\rangle \left( |00\rangle + |11\rangle \right) + \frac{\beta}{\sqrt{2}} |1\rangle \left( |10\rangle + |01\rangle \right)
\]

\[
\xrightarrow{H \otimes I \otimes I} \frac{\alpha}{2} \left( |0\rangle + |1\rangle \right) \left( |00\rangle + |11\rangle \right) + \frac{\beta}{2} \left( |0\rangle - |1\rangle \right) \left( |10\rangle + |01\rangle \right)
\]

\[
= \frac{1}{2} |00\rangle \left( \alpha |0\rangle + \beta |1\rangle \right) \quad \text{Alice measures $00$ so Bob applies $I$}
\]

\[
+ \frac{1}{2} |01\rangle \left( \alpha |1\rangle + \beta |0\rangle \right) \quad \text{Alice measures $01$ so Bob applies $X$}
\]

\[
+ \frac{1}{2} |10\rangle \left( \alpha |0\rangle - \beta |1\rangle \right) \quad \text{Alice measures $10$ so Bob applies $Z$}
\]

\[
+ \frac{1}{2} |11\rangle \left( \alpha |1\rangle - \beta |0\rangle \right) \quad \text{Alice measures $11$ so Bob applies $ZX$}
\]
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Test of entanglement

Two isolated parties Alice and Bob

- Alice gets coin toss $x$, replies $a$
- Bob gets coin toss $y$, replies $b$

Goal: maximize $a \oplus b = x \land y$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x \land y$</th>
<th>$a \oplus b$</th>
<th>winning options for $(a, b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(0,0) or (1,1)</td>
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<tr>
<td>0</td>
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<td>(0,0) or (1,1)</td>
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</tr>
</tbody>
</table>
Best classical strategy to maximize $a \oplus b = x \land y$

Proof that any assignment to $a$ and $b$ cannot always satisfy $a \oplus b = x \land y$

1. Let $a_0$ be Alice’s response if she sees $x = 0$
2. Let $a_1$ be Alice’s response if she sees $x = 1$
3. Let $b_0$ be Bob’s response if she sees $y = 0$
4. Let $b_1$ be Bob’s response if she sees $y = 1$

Satisfy $a \oplus b = x \land y$

1. $a_0 \oplus b_0 = 0$
2. $a_0 \oplus b_1 = 0$
3. $a_1 \oplus b_0 = 0$
4. $a_1 \oplus b_1 = 1$

Sum (mod 2) of left side

$$(a_0 \oplus b_0) \oplus (a_0 \oplus b_1) \oplus (a_1 \oplus b_0) \oplus (a_1 \oplus b_1) = 0$$

Sum (mod 2) of right side

1
Best classical strategy to maximize $a \oplus b = x \land y$

Even if the two shared randomness, the random coin toss of $x$ and $y$ prevents use of shared randomness. Best you can do is $3/4$. Give a couple ways of getting $3/4$
A quantum strategy to maximize $a \oplus b = x \wedge y$

Alice and Bob share entangled pair $|\Phi\rangle$

$$|\Phi\rangle = \frac{1}{\sqrt{12}} \left( 3 |00\rangle + |01\rangle + |10\rangle - |11\rangle \right)$$

$(x, y) = (0, 0)$ So Alice and Bob both apply $I$:

$$(I \otimes I) |\Phi\rangle = \frac{1}{\sqrt{12}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Measurement yields

$$\begin{cases} (a, b) = (0, 0), \text{ a win, with probability } \frac{9}{12} \\ (a, b) = (0, 1), \text{ a loss, with probability } \frac{1}{12} \\ (a, b) = (1, 0), \text{ a loss, with probability } \frac{1}{12} \\ (a, b) = (1, 1), \text{ a win, with probability } \frac{1}{12} \end{cases}$$
A quantum strategy to maximize $a \oplus b = x \land y$

Alice and Bob share entangled pair $|\Phi\rangle$

$$|\Phi\rangle = \frac{1}{\sqrt{12}} \left(3 |00\rangle + |01\rangle + |10\rangle - |11\rangle \right)$$

$(x, y) = (0, 1)$ So Alice applies $I$, Bob applies $H$: 

$$(I \otimes H) |\Phi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \frac{1}{\sqrt{12}} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2\sqrt{6}} \begin{bmatrix} 4 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

Measurement yields

$$\begin{cases} (a, b) = (0, 0), \text{ a win, with probability } \frac{4}{6} \\ (a, b) = (0, 1), \text{ a loss, with probability } \frac{1}{6} \\ (a, b) = (1, 0), \text{ a loss, with probability } 0 \\ (a, b) = (1, 1), \text{ a win, with probability } \frac{1}{6} \end{cases}$$
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Alice and Bob share entangled pair $|\Phi\rangle$

$$|\Phi\rangle = \frac{1}{\sqrt{12}} \left( 3 |00\rangle + |01\rangle + |10\rangle - |11\rangle \right)$$

$(x, y) = (1, 0)$ So Alice applies $H$, Bob applies $I$:

$$(H \otimes I) |\Phi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \frac{1}{\sqrt{12}} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2\sqrt{6}} \begin{bmatrix} 4 \\ 0 \\ 2 \\ 2 \end{bmatrix}$$

Measurement yields

$$\begin{cases} (a, b) = (0, 0), \text{ a win, with probability } \frac{4}{6} \\ (a, b) = (0, 1), \text{ a loss, with probability } 0 \\ (a, b) = (1, 0), \text{ a loss, with probability } \frac{1}{6} \\ (a, b) = (1, 1), \text{ a win, with probability } \frac{1}{6} \end{cases}$$
A quantum strategy to maximize $a \oplus b = x \land y$

Alice and Bob share entangled pair $|\Phi\rangle$

$|\Phi\rangle = \frac{1}{\sqrt{12}} \left( 3 |00\rangle + |01\rangle + |10\rangle - |11\rangle \right)$

$(x, y) = (1, 1)$ So Alice and Bob both apply $H$:

$$(H \otimes H) |\Phi\rangle = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \frac{1}{\sqrt{12}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{4\sqrt{3}} \begin{bmatrix} 4 \\ 4 \\ 4 \\ 0 \end{bmatrix}$$

Measurement yields

$$\begin{cases} (a, b) = (0, 0), \text{ a loss, with probability } \frac{1}{3} \\ (a, b) = (0, 1), \text{ a win, with probability } \frac{1}{3} \\ (a, b) = (1, 0), \text{ a win, with probability } \frac{1}{3} \\ (a, b) = (1, 1), \text{ a loss, with probability } 0 \end{cases}$$