Languages and representations for quantum computing: Stabilizer formalism

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What is it that gives quantum computers an advantage compared to classical computing?

- Superposition?
- Entanglement?
- Both?
- Neither?
Importance of representations in quantum intuition, programming, and simulation

- Conventional quantum circuits and state vector view of QC conceals symmetries, hinders intuition.
- Classical simulation of quantum computing is actually tractable for a certain subset of quantum gates.
- Both the logical and native gatesets in a quantum architecture need to be universal for quantum advantage.
Several views/representations of quantum computing

- Programming has several views: functional programming, procedural programming.
- Physics has several views: Newtonian, Lagrangian, Hamiltonian

Different views reveal different symmetries, offer different intuition.
Several views/representations of quantum computing

- Schrödinger: state vectors and density matrices
- Heisenberg: stabilizer formalism
- Tensor-network
- Feynman: path sums

A survey of these representations of quantum computing is given in Chapter 9 of this recent book [Ding and Chong, 2020].
Several views/representations of quantum computing

- Schrödinger: state vectors and density matrices
- Heisenberg: stabilizer formalism
- Tensor-network
- Feynman: path sums
- Binary decision diagrams (new?)
- Logical satisfiability equations
Schrödinger view

- In Schrödinger quantum mechanics description, emphasis on how states evolve.

\[ \text{CNOT}_{0,1}(H_0 \otimes I_1) |00\rangle = \text{CNOT}_{0,1} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

- The Schrödinger view requires exponential storage: A quantum computer with N qubits can be in superposition of \(2^N\) basis states, requires \(2^N\) amplitudes to fully specify state.
In Heisenberg quantum mechanics description, emphasis on how operators evolve.

If we limit operations to the Clifford gates (a subset of quantum gates), simulation tractable in polynomial time and space.

Covers some quantum algorithms: quantum superdense coding, quantum teleportation, Deutsch-Jozsa, Bernstein-Vazirani, quantum error correction, most quantum error correction protocols.

A model for probabilistic (but not quantum) computation.
Concrete example on Bell state circuit

\[ CNOT_{0,1}(H_0 \otimes I_1) |00\rangle \]

1. Start with N qubits with initial state \(|0\rangle \otimes N\).
2. Represent the state as its group of stabilizers—\(|00\rangle : \{IZ, ZI\}\)
3. When simulating the quantum circuit, decompose the Clifford gates to stabilizer gates \{CNOT, H, P\}.
4. Apply each of the stabilizer gates to the stabilizer representation.
   - Hadamard on first qubit—\(|+\rangle |0\rangle : \{IZ, XI\}\)
   - CNOT on both qubits—\(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\) : \{ZZ, XX\}
Representing a state as its group of stabilizers

- A unitary operator $U$ stabilizes a pure state $|\psi\rangle$ if $U|\psi\rangle = |\psi\rangle$
Representing a state as its group of stabilizers

1. $I$ stabilizes everything.
2. $-I$ stabilizes nothing.

3. $X$ stabilizes $|+\rangle$: $X|+\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |+\rangle$

4. $-X$ stabilizes $|-\rangle$: $-X|-\rangle = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |-\rangle$

5. $Y$ stabilizes $|+i\rangle$: $Y|+i\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |+i\rangle$

6. $-Y$ stabilizes $|-i\rangle$: $-Y|-i\rangle = -\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |-i\rangle$

7. $Z$ stabilizes $|0\rangle$: $Z|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$

8. $-Z$ stabilizes $|1\rangle$: $-Z|1\rangle = -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$
In other words,

1. $|0\rangle$ is stabilized by $\{I, Z\}$
2. $|1\rangle$ is stabilized by $\{I, -Z\}$
3. $|+\rangle$ is stabilized by $\{I, X\}$
4. $|-\rangle$ is stabilized by $\{I, -X\}$
5. $|+i\rangle$ is stabilized by $\{I, Y\}$
6. $|-i\rangle$ is stabilized by $\{I, -Y\}$
Special places on the Bloch sphere

\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \]
\[ = |\alpha|[\cos(\gamma) + i \cdot \sin(\gamma)] |0\rangle \]
\[ + |\beta|[\cos(\gamma + \phi) + i \cdot \sin(\gamma + \phi)] |1\rangle \]
\[ = \cos(\frac{\theta}{2})e^{i\gamma} |0\rangle + \sin(\frac{\theta}{2})e^{i(\gamma+\phi)} |1\rangle \]

Enforces \( |\alpha|^2 + |\beta|^2 = 1 \)

Figure: Bloch sphere showing pole states.
Source: Wikimedia.
Special places on the Bloch sphere

\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \]
\[ = \alpha | [\cos(\gamma) + i \cdot \sin(\gamma)] |0\rangle \]
\[ + \beta | [\cos(\gamma + \phi) + i \cdot \sin(\gamma + \phi)] |1\rangle \]
\[ = \cos\left(\frac{\theta}{2}\right)e^{i\gamma} |0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i(\gamma+\phi)} |1\rangle \]

Enforces \( |\alpha|^2 + |\beta|^2 = 1 \)

**Figure:** Bloch sphere showing pole states.
Source: Wikimedia.
Representing a state as its group of stabilizers

For multi-qubit states, the group of stabilizers is the cartesian product of the single-qubit stabilizers

- $|00\rangle = |0\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, Z \otimes I, Z \otimes Z\}$
- $\frac{|00\rangle + |10\rangle}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle = |+\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, X \otimes I, X \otimes Z\}$
- $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ is stabilized by $\{I \otimes I, X \otimes X, -Y \otimes Y, Z \otimes Z\}$
Representing a state as its group of stabilizers

- Critical result from group theory: for any N-qubit stabilized state, only N elements needed to specify group—a result from abstract algebra group theory [Nielsen and Chuang, 2002, Appendix 2]
- So long as the quantum circuit consists only of Clifford gates, only N elements needed to specify whole quantum state.
- Contrast against $2^N$ amplitudes needed to specify a general N-qubit quantum state vector.
- For example a two-qubit states needs four amplitudes $\{a_0, a_1, a_2, a_3\}$ to specify quantum state $|\psi\rangle = a_0 |00\rangle + a_1 |01\rangle + a_2 |10\rangle + a_3 |11\rangle$. 
Representing a state as its group of stabilizers

Critical result from group theory: for any N-qubit stabilized state, only N elements needed to specify group.

1. $\left| 0 \right\rangle$ is stabilized by $\{I, Z\}$, $Z$ is generator
2. $\left| 1 \right\rangle$ is stabilized by $\{I, -Z\}$, $-Z$ is generator
3. $\left| + \right\rangle$ is stabilized by $\{I, X\}$, $X$ is generator
4. $\left| - \right\rangle$ is stabilized by $\{I, -X\}$, $-X$ is generator
5. $\left| +i \right\rangle$ is stabilized by $\{I, Y\}$, $Y$ is generator
6. $\left| -i \right\rangle$ is stabilized by $\{I, -Y\}$, $-Y$ is generator
7. $\left| 0 \right\rangle \otimes \left| 0 \right\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, Z \otimes I, Z \otimes Z\}$, $\{I \otimes Z, Z \otimes I\}$ is generator
8. $\left| + \right\rangle \otimes \left| 0 \right\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, X \otimes I, X \otimes Z\}$, $\{I \otimes Z, X \otimes I\}$ is generator
9. $\frac{\left| 00 \right\rangle + \left| 11 \right\rangle}{\sqrt{2}}$ is stabilized by $\{I \otimes I, X \otimes X, -Y \otimes Y, X \otimes Z\}$, $\{X \otimes X, Z \otimes Z\}$ is generator
Concrete example on Bell state circuit

\[ CNOT_{0,1}(H_0 \otimes I_1) \ket{00} \]

1. Start with \( N \) qubits with initial state \( \ket{0}^\otimes N \).
2. Represent the state as its group of stabilizers—\( \ket{00} : \{IZ, ZI\} \)
3. When simulating the quantum circuit, decompose the Clifford gates to stabilizer gates \( \{CNOT, H, P\} \).
4. Apply each of the stabilizer gates to the stabilizer representation.
   - Hadamard on first qubit—\( \ket{+} \ket{0} \): \( \{IZ, XI\} \)
   - CNOT on both qubits—\( \frac{\ket{00} + \ket{11}}{\sqrt{2}} \): \( \{ZZ, XX\} \)
Stabilizer gates: \{CNOT, H, P\}

2. CNOT gate: induces entanglement.
3. Phase gate: induces complex phases. \( P = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \)

- Despite featuring superposition, entanglement, and complex amplitudes, is \textit{not} universal for quantum computing.
- We shall see that the deeply symmetrical structure of these gates prevent access to full quantum Hilbert space.
Stabilizer gates are a generator for Pauli gates (i.e., Clifford gates decompose to stabilizer gates)

Pauli gates are rotations around respective axes by $\pi$.

- $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = PP$

- $X = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = HZH$

- $Y = iXZ$

- $X^2 = Y^2 = Z^2 = I$

- Symmetry is similar to quaternions.

- With Clifford gates consisting of \{\text{CNOT, } H, P, I, X, Y, Z\}, sufficient to build many quantum algorithms, including: quantum superdense coding, quantum teleportation, Deutsch-Jozsa, Bernstein-Vazirani, quantum error correction, most quantum error correction protocols.
Single qubit stabilizer gates bounce stabilizer states around an octahedron on the Bloch sphere

\[
P |0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle
\]

\[
P |1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = i |1\rangle
\]

\[
P |+\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{2}} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{i} \\ \sqrt{2} \end{bmatrix} = |+i\rangle
\]

\[
P |−\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{2}} \\ -\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{-i} \\ -\sqrt{2} \end{bmatrix} = |−i\rangle
\]

\[
P |+i\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{i} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\sqrt{2}}{\sqrt{2}} \end{bmatrix} = |−\rangle
\]

\[
P |−i\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{-i} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\sqrt{2}}{-1} \end{bmatrix} = |+\rangle
\]
Single qubit stabilizer gates bounce stabilizer states around an octahedron on the Bloch sphere

$$H |0\rangle = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |+\rangle$$

$$H |1\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = |-\rangle$$

$$H |+\rangle = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

$$H |-\rangle = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

$$H |+i\rangle = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{i} \\ \frac{-1}{i} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{-i} \\ \frac{-1}{i} \end{bmatrix} = |i\rangle$$

$$H |-i\rangle = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = |+i\rangle$$
Apply each of the stabilizer gates to the stabilizer representation.

- **Phase:**
  1. $Z \rightarrow Z$
  2. $-Z \rightarrow -Z$
  3. $X \rightarrow Y$
  4. $-X \rightarrow -Y$
  5. $Y \rightarrow -X$
  6. $-Y \rightarrow X$

- **Hadamard:**
  1. $Z \rightarrow X$
  2. $-Z \rightarrow -X$
  3. $X \rightarrow Z$
  4. $-X \rightarrow -Z$
  5. $Y \rightarrow -Y$
  6. $-Y \rightarrow Y$
Single qubit stabilizer gates bounce stabilizer states around an octahedron on the Bloch sphere

Figure: Bloch sphere showing pole states. Source: Wikimedia.
Apply each of the stabilizer gates to the stabilizer representation.

- **CNOT:**
  1. $X \otimes I \rightarrow X \otimes X$
  2. $I \otimes X \rightarrow I \otimes X$
  3. $Z \otimes I \rightarrow Z \otimes I$
  4. $I \otimes Z \rightarrow Z \otimes Z$
Concrete example on Bell state circuit

\[ CNOT_{0,1}(H_0 \otimes I_1) |00\rangle \]

1. Start with \( N \) qubits with initial state \(|0\rangle^\otimes N\).
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   - CNOT on both qubits—\( \frac{|00\rangle + |11\rangle}{\sqrt{2}} \): \( \{ZZ, XX\} \)
Gottesman-Knill theorem states that there exists a classical algorithm that simulates any stabilizer circuit in polynomial time.

Any quantum state created by a Clifford circuit, even if it has lots of superpositions and entanglement, is easy to classically simulate.

Quantum computers need at least one non-Clifford gate to achieve universal quantum computation.

The T gate, where $TT = P$, $PP = Z$ is one common choice.

There are results showing that a quantum circuit is only exponentially hard to simulate w.r.t. the number of T-gates.
References

- Main sources: [Gottesman, 1998] [Aaronson, ]
- Further reference on separation of probabilistic and quantum computing: [Van Den Nes, 2010]
- Further reference on applications in classical simulation of Clifford quantum circuits: [Aaronson and Gottesman, 2004]
- Further reference on applications in classical simulation of general quantum circuits: [Bravyi and Gosset, 2016]
References

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Improved simulation of stabilizer circuits.

Improved classical simulation of quantum circuits dominated by clifford gates.

Quantum computer systems: Research for noisy intermediate-scale quantum computers.

The heisenberg representation of quantum computers.

Quantum computation and quantum information.

Classical simulation of quantum computation, the gottesman-knill theorem, and slightly beyond.