Languages and representations for quantum computing: Stabilizer formalism

Yipeng Huang

Rutgers University

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What is it that gives quantum computers an advantage compared to classical computing?

- Superposition?
- Entanglement?
- Both?
- Neither?

Importance of representations in quantum intuition, programming, and simulation

- Conventional quantum circuits and state vector view of QC conceals symmetries, hinders intuition.
- Classical simulation of quantum computing is actually tractable for *a certain subset* of quantum gates.
- Both the logical and native gatesets in a quantum architecture need to be *universal* for quantum advantage.

Several views/representations of quantum computing

- Programming has several views: functional programming, procedural programming.
- Physics has several views: Newtonian, Lagrangian, Hamiltonian

Different views reveal different symmetries, offer different intuition.

Several views/representations of quantum computing

- Schrödinger: state vectors and density matrices
- Heisenberg: stabilizer formalism
- Tensor-network
- Feynman: path sums

A survey of these representations of quantum computing is given in Chapter 9 of this recent book [Ding and Chong, 2020].

Several views/representations of quantum computing

- Schrödinger: state vectors and density matrices
- Heisenberg: stabilizer formalism
- Tensor-network
- Feynman: path sums
- Binary decision diagrams (new?)
- Logical satisfiability equations

Schrödinger view

- In Schrödinger quantum mechanics description, emphasis on how *states* evolve.
- $CNOT_{0,1}(H_0 \otimes I_1)\,|00\rangle = CNOT_{0,1} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} =$
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$
- The Schrodinger view requires exponential storage: A quantum computer with N qubits can be in superposition of 2^N basis states, requires 2^N amplitudes to fully specify state.

Heisenberg view / stabilizer formalism

- In Heisenberg quantum mechanics description, emphasis on how *operators* evolve.
- If we limit operations to the Clifford gates (a subset of quantum gates), simulation tractable in polynomial time and space.
- Covers some quantum algorithms: quantum superdense coding, quantum teleportation, Deutsch-Jozsa, Bernstein-Vazirani, quantum error correction, most quantum error correction protocols.
- A model for probabilistic (but not quantum) computation.

Concrete example on Bell state circuit

$$CNOT_{0,1}(H_0 \otimes I_1)\,|00\rangle$$

1. Start with N qubits with initial state $|0\rangle^{\otimes N}$.
2. Represent the state as its group of stabilizers—$|00\rangle : \{IZ, ZI\}$
3. When simulating the quantum circuit, decompose the Clifford gates to stabilizer gates $\{CNOT, H, P\}$.
4. Apply each of the stabilizer gates to the stabilizer representation.

- Hadamard on first qubit—$|+\rangle\,|0\rangle$: $\{IZ, XI\}$
- CNOT on both qubits—$\frac{|00\rangle+|11\rangle}{\sqrt{2}}$: $\{ZZ, XX\}$

Representing a state as its group of stabilizers

- A unitary operator U stabilizes a pure state $|\psi\rangle$ if $U|\psi\rangle = |\psi\rangle$

Representing a state as its group of stabilizers

1. I stabilizes everything.
2. $-I$ stabilizes nothing.
3. X stabilizes $|+\rangle$: $X|+\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |+\rangle$
4. $-X$ stabilizes $|-\rangle$: $-X|-\rangle = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = |-\rangle$
5. Y stabilizes $|+i\rangle$: $Y|+i\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = |+i\rangle$
6. $-Y$ stabilizes $|-i\rangle$: $-Y|-i\rangle = -\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix} = |-i\rangle$
7. Z stabilizes $|0\rangle$: $Z|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$
8. $-Z$ stabilizes $|1\rangle$: $-Z|1\rangle = -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$

Representing a state as its group of stabilizers

In other words,

1. $|0\rangle$ is stabilized by $\{I, Z\}$
2. $|1\rangle$ is stabilized by $\{I, -Z\}$
3. $|+\rangle$ is stabilized by $\{I, X\}$
4. $|-\rangle$ is stabilized by $\{I, -X\}$
5. $|+i\rangle$ is stabilized by $\{I, Y\}$
6. $|-i\rangle$ is stabilized by $\{I, -Y\}$

Special places on the Bloch sphere

$$
\begin{aligned}
|\psi\rangle &= \alpha\,|0\rangle + \beta\,|1\rangle \\
&= |\alpha|[cos(\gamma) + i \cdot sin(\gamma)]\,|0\rangle \\
&+ |\beta|[cos(\gamma+\phi) + i \cdot sin(\gamma+\phi)]\,|1\rangle \\
&= cos(\frac{\theta}{2})e^{i\gamma}\,|0\rangle + sin(\frac{\theta}{2})e^{i(\gamma+\phi)}\,|1\rangle
\end{aligned}
$$

Enforces $|\alpha|^2 + |\beta|^2 = 1$

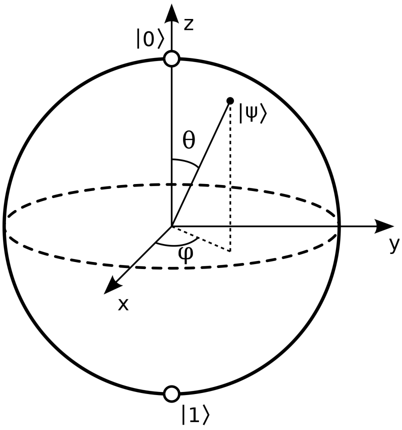

Figure: Bloch sphere showing pole states. Source: Wikimedia.

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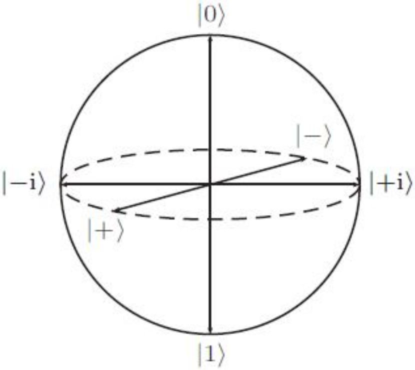

Figure: Bloch sphere showing pole states. Source: Wikimedia.

Representing a state as its group of stabilizers

For multi-qubit states, the group of stabilizers is the cartesian product of the single-qubit stabilizers

- $|00\rangle = |0\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, Z \otimes I, Z \otimes Z\}$
- $\frac{|00\rangle+|10\rangle}{\sqrt{2}} = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes |0\rangle = |+\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, X \otimes I, X \otimes Z\}$
- $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ is stabilized by $\{I \otimes I, X \otimes X, -Y \otimes Y, Z \otimes Z\}$

Representing a state as its group of stabilizers

- Critical result from group theory: for any N-qubit stabilized state, only N elements needed to specify group—a result from abstract algebra group theory [Nielsen and Chuang, 2002, Appendix 2]
- So long as the quantum circuit consists only of Clifford gates, only N elements needed to specify whole quantum state.
- Contrast against 2^N amplitudes needed to specify a general N-qubit quantum state vector.
- For example a two-qubit states needs four amplitues $\{a_0, a_1, a_2, a_3\}$ to specify quantum state $|\psi\rangle = a_0 |00\rangle + a_1 |01\rangle + a_2 |10\rangle + a_3 |11\rangle$.

Representing a state as its group of stabilizers

Critical result from group theory: for any N-qubit stabilized state, only N elements needed to specify group.

1. $|0\rangle$ is stabilized by $\{I, Z\}$, Z is generator
2. $|1\rangle$ is stabilized by $\{I, -Z\}$, -Z is generator
3. $|+\rangle$ is stabilized by $\{I, X\}$, X is generator
4. $|-\rangle$ is stabilized by $\{I, -X\}$, -X is generator
5. $|+i\rangle$ is stabilized by $\{I, Y\}$, Y is generator
6. $|-i\rangle$ is stabilized by $\{I, -Y\}$, -Y is generator
7. $|0\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, Z \otimes I, Z \otimes Z\}$, $\{I \otimes Z, Z \otimes I\}$ is generator
8. $|+\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, X \otimes I, X \otimes Z\}$, $\{I \otimes Z, X \otimes I\}$ is generator
9. $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ is stabilized by $\{I \otimes I, X \otimes X, -Y \otimes Y, X \otimes Z\}$, $\{X \otimes X, Z \otimes Z\}$ is generator

Concrete example on Bell state circuit

$$CNOT_{0,1}(H_0 \otimes I_1)\left|00\right\rangle$$

1. Start with N qubits with initial state $\left|0\right\rangle^{\otimes N}$.
2. Represent the state as its group of stabilizers—$\left|00\right\rangle : \{IZ, ZI\}$
3. When simulating the quantum circuit, decompose the Clifford gates to stabilizer gates $\{CNOT, H, P\}$.
4. Apply each of the stabilizer gates to the stabilizer representation.

- Hadamard on first qubit—$\left|+\right\rangle\left|0\right\rangle$: $\{IZ, XI\}$
- CNOT on both qubits—$\frac{\left|00\right\rangle+\left|11\right\rangle}{\sqrt{2}}$: $\{ZZ, XX\}$

Stabilizer gates: $\{CNOT, H, P\}$

1. Hadamard gate: induces superpositions.
2. CNOT gate: induces entanglement.
3. Phase gate: induces complex phases. $P = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

- Despite featuring superposition, entanglement, and complex amplitudes, is *not* universal for quantum computing.
- We shall see that the deeply symmetrical structure of these gates prevent access to full quantum Hilbert space.

Stabilizer gates are a generator for Pauli gates (i.e., Clifford gates decompose to stabilizer gates)

Pauli gates are rotations around respective axes by π.

- $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = PP$
- $X = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = HZH$
- $Y = iXZ$
- $X^2 = Y^2 = Z^2 = I$
- Symmetry is similar to quaternions.
- With Clifford gates consisting of $\{CNOT, H, P, I, X, Y, Z\}$, sufficient to build many quantum algorithms, including: quantum superdense coding, quantum teleportation, Deutsch-Jozsa, Bernstein-Vazirani, quantum error correction, most quantum error correction protocols.

Single qubit stabilizer gates bounce stabilizer states around an octahedron on the Bloch sphere

$$P|0\rangle = \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix} = |0\rangle$$

$$P|1\rangle = \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix} = i|1\rangle$$

$$P|+\rangle = \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{bmatrix} = |+i\rangle$$

$$P|-\rangle = \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-i}{\sqrt{2}}\end{bmatrix} = |-i\rangle$$

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$$P|-i\rangle = \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-i}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = |+\rangle$$

Single qubit stabilizer gates bounce stabilizer states around an octahedron on the Bloch sphere

$$H|0\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |+\rangle$$

$$H|1\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = |-\rangle$$

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Apply each of the stabilizer gates to the stabilizer representation.

- Phase:
  1. $Z \to Z$
  2. $-Z \to -Z$
  3. $X \to Y$
  4. $-X \to -Y$
  5. $Y \to -X$
  6. $-Y \to X$
- Hadamard:
  1. $Z \to X$
  2. $-Z \to -X$
  3. $X \to Z$
  4. $-X \to -Z$
  5. $Y \to -Y$
  6. $-Y \to Y$

Single qubit stabilizer gates bounce stabilizer states around an octahedron on the Bloch sphere

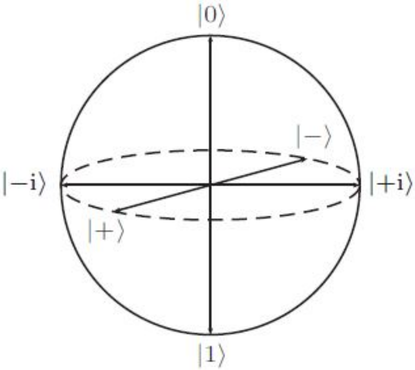

Figure: Bloch sphere showing pole states. Source: Wikimedia.

Apply each of the stabilizer gates to the stabilizer representation.

- CNOT:
  1. $X \otimes I \rightarrow X \otimes X$
  2. $I \otimes X \rightarrow I \otimes X$
  3. $Z \otimes I \rightarrow Z \otimes I$
  4. $I \otimes Z \rightarrow Z \otimes Z$

Concrete example on Bell state circuit

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Gottesman-Knill theorem and its implications

- Gottesman-Knill theorem states that there exists a classical algorithm that simuates any stabilizer circuit in polynomial time.
- Any quantum state created by a Clifford circuit, even if it has lots of superpositions and entanglement, is easy to classically simulate.
- Quantum computers need at least one non-Clifford gate to achieve universal quantum computation.
- The T gate, where $TT = P, PP = Z$ is one common choice.
- There are results showing that a quantum circuit is only exponentially hard to simulate w.r.t. the number of T-gates.

References

- Main sources: [Gottesman, 1998] [Aaronson,]
- Further reference on separation of probabilistic and quantum computing: [Van Den Nes, 2010]
- Further reference on applications in classical simulation of Clifford quantum circuits: [Aaronson and Gottesman, 2004]
- Further reference on applications in classical simulation of general quantum circuits: [Bravyi and Gosset, 2016]

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