Emerging languages and representations for quantum computing: Tensor networks

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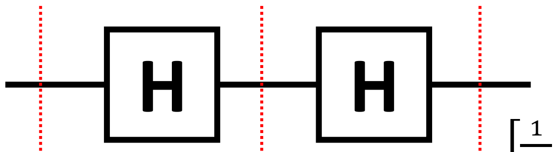
Significance of stabilizers and tensor networks

- Highlight connections between quantum computing semantics to other areas of computer science.
  - Turns quantum computing from an analytical, numerical field into one that has ties to algebra and topology
- Introduce these two important alternative representations for quantum computing
  - Heisenberg view (stabilizer formalism) is important in quantum error correction literature
  - Feynman view (path sums and tensor network contraction) is important in quantum circuit simulation literature

Schrödinger view of quantum computing



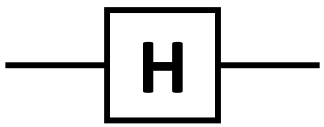
$$\begin{bmatrix}0\\1\end{bmatrix}$$

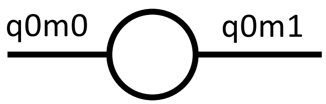
$$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}$$

$$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$

Emphasis is on finding state vectors at each moment.

Feynman (path sum) view of quantum computing





A tensor network

	-
q0m0 =$\|0\rangle$	$\begin{bmatrix}0\\1\end{bmatrix}$
q0m0 =$\|1\rangle$	

	q0m0 =$\|0\rangle$	q0m0 =$\|1\rangle$
q0m1 =$\|0\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
q0m1 =$\|1\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$

q0m0	q0m1	
$\|0\rangle$	$\|0\rangle$	$0 \bullet \frac{1}{\sqrt{2}} = 0$
$\|0\rangle$	$\|1\rangle$	$0 \bullet \frac{1}{\sqrt{2}} = 0$
$\|1\rangle$	$\|0\rangle$	$1 \bullet \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$
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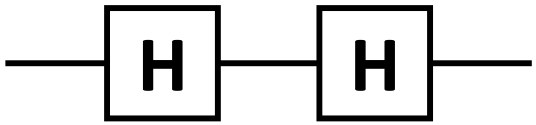
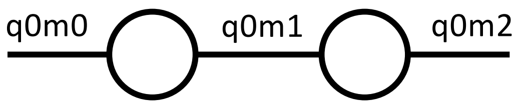
Feynman (path sum) view of quantum computing





A tensor network

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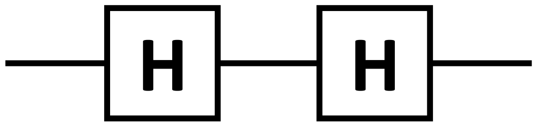
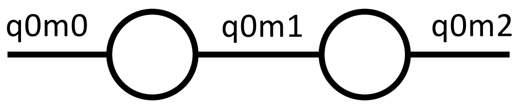
	q0m0 =$\|0\rangle$	**q0m0 =$\|1\rangle$**
q0m1 =$\|0\rangle$	$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}$	
q0m1 =$\|1\rangle$		

	q0m1 =$\|0\rangle$	**q0m1 =$\|1\rangle$**
q0m2 =$\|0\rangle$	$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}$	
q0m2 =$\|1\rangle$		

Feynman (path sum) view of quantum computing





A tensor network

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q0m1 =$\|1\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$

	q0m1 =$\|0\rangle$	q0m1 =$\|1\rangle$
q0m2 =$\|0\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
q0m2 =$\|1\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$

q0m0	q0m1	q0m2	
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = 0$
$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = 0$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = 0$
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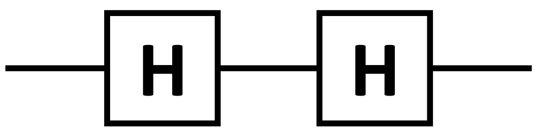
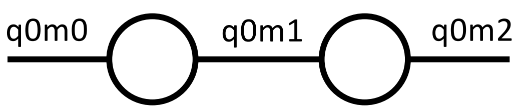
Feynman (path sum) view of quantum computing





A tensor network

1. In this calculation technique, we don't form the whole state vector.
2. Path sum for q0m2=|0> has destructive interference.

	-
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q0m0 $=\|1\rangle$	

	q0m0 $=\|0\rangle$	q0m0 $=\|1\rangle$
q0m1 $=\|0\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
q0m1 $=\|1\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$

	q0m1 $=\|0\rangle$	q0m1 $=\|1\rangle$
q0m2 $=\|0\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
q0m2 $=\|1\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$

q0m0	q0m1	q0m2	
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = 0$
$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = 0$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = 0$
$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{-1}{\sqrt{2}} = 0$
$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$1 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = \frac{1}{2}$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$1 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = \frac{1}{2}$
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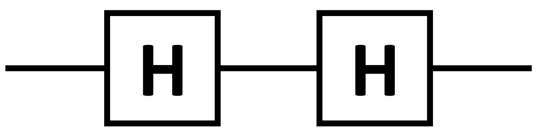
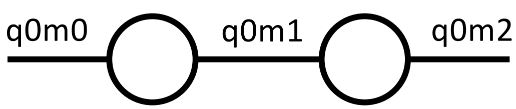
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$1 \bullet \frac{-1}{\sqrt{2}} \bullet \frac{-1}{\sqrt{2}} = \frac{1}{2}$

Feynman (path sum) view of quantum computing





A tensor network

1. In this calculation technique, we don't form the whole state vector.
2. Path sum for q0m2=|0> has destructive interference.
3. Path sum for q0m2=|1> has constructive interference.

	-
q0m0 $=\|0\rangle$	$\begin{bmatrix}0\\1\end{bmatrix}$
q0m0 $=\|1\rangle$	

	q0m0 $=\|0\rangle$	q0m0 $=\|1\rangle$
q0m1 $=\|0\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
q0m1 $=\|1\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$

	q0m1 $=\|0\rangle$	q0m1 $=\|1\rangle$
q0m2 $=\|0\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
q0m2 $=\|1\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$

q0m0	q0m1	q0m2	
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = 0$
$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = 0$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = 0$
$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$0 \bullet \frac{1}{\sqrt{2}} \bullet \frac{-1}{\sqrt{2}} = 0$
$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$1 \bullet \frac{1}{\sqrt{2}} \bullet \frac{1}{\sqrt{2}} = \frac{1}{2}$
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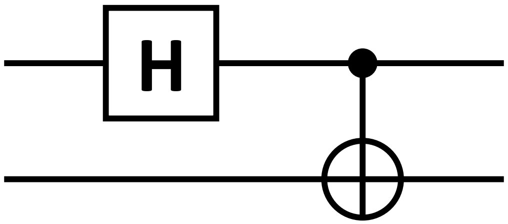
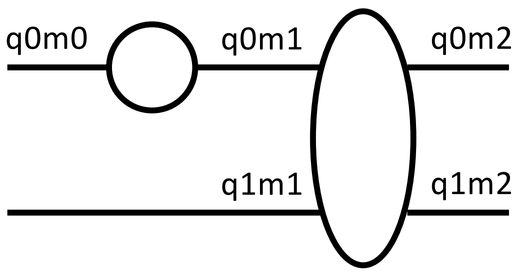
Quantum circuits are a specialized form of tensor network.

- Tensor: a data structure with rank k and dimension m.
  - Rank-0 tensor: scalar.
  - Rank-1 tensor: vector.
  - Rank-2 tensor: matrix.
  - Rank-3 tensor…
- For qubits, dimension m = 2.
  - Rank-1 tensor: a single qubit state.
  - Rank-2 tensor: a 2x2 matrix for a single-qubit gate.
  - Rank-4 tensor: a 2x2x2x2 data structure for a two-qubit gate.





A rank-1 tensor

	-
q0m0 $=\vert 0\rangle$	$\begin{bmatrix}0\\1\end{bmatrix}$
q0m0 $=\vert 1\rangle$	

A rank-2 tensor

	q0m0 $=\vert 0\rangle$	**q0m0 $=\vert 1\rangle$**
q0m1 $=\vert 0\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
q0m1 $=\vert 1\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$

A rank-4 tensor

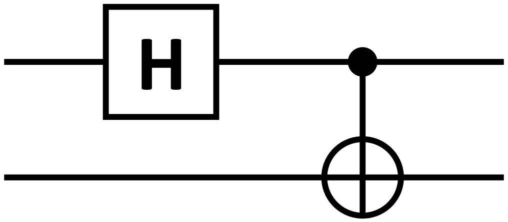
		q0m1 $=\vert 0\rangle$		**q0m1 $=\vert 1\rangle$**	
		q1m1 $=\vert 0\rangle$	**q1m1 $=\vert 1\rangle$**	**q1m1 $=\vert 0\rangle$**	**q1m1 $=\vert 1\rangle$**
q0m2 $=\vert 0\rangle$	**q1m2 $=\vert 0\rangle$**	1	0	0	0
	q1m2 $=\vert 1\rangle$	0	1	0	0
q0m2 $=\vert 1\rangle$	**q1m2 $=\vert 0\rangle$**	0	0	0	1
	q1m2 $=\vert 1\rangle$	0	0	1	0

Tensor network contraction

- Tensor network contraction is one type of tensor-tensor multiplication. It is a generalized form of matrix multiplication.
- Merge two tensors into one. Absorb common edges. If the two tensors share a common index, sum over all possible values of that index.
- For example: contract tensor A and tensor B into tensor C.
  - Where A has rank (x+y), B has rank (y+z), C has rank (x+z)

$$C_{i_1,i_2\ldots,i_x,k_1,k_2\ldots,k_z} = \sum_{j_1,j_2\ldots,j_y \in \{0,..,m-1\}} A_{i_1,i_2\ldots,i_x,j_1,\ldots,j_y} B_{j_1,j_2\ldots,j_y,k_1,\ldots,k_z}$$

Credit: Fried et al. qTorch: The quantum tensor contraction handler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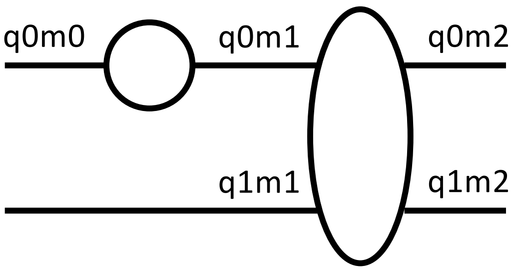
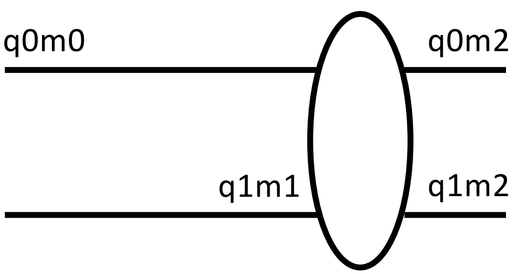

q0m0
q0m1
q0m2
q1m1
q1m2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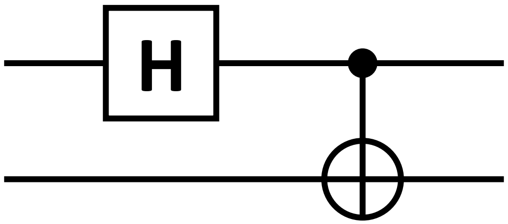
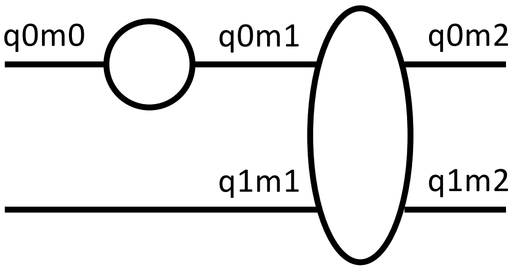
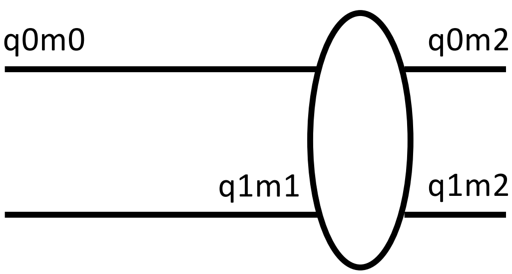


	q0m0 $=\|0\rangle$	q0m0 $=\|1\rangle$
q0m1 $=\|0\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
q0m1 $=\|1\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$



		q0m1 $=\|0\rangle$		q0m1 $=\|1\rangle$	
		q1m1 $=\|0\rangle$	q1m1 $=\|1\rangle$	q1m1 $=\|0\rangle$	q1m1 $=\|1\rangle$
q0m2 $=\|0\rangle$	q1m2 $=\|0\rangle$	1	0	0	0
	q1m2 $=\|1\rangle$	0	1	0	0
q0m2 $=\|1\rangle$	q1m2 $=\|0\rangle$	0	0	0	1
	q1m2 $=\|1\rangle$	0	0	1	0



		q0m0 $=\|0\rangle$		q0m0 $=\|1\rangle$	
		q1m1 $=\|0\rangle$	q1m1 $=\|1\rangle$	q1m1 $=\|0\rangle$	q1m1 $=\|1\rangle$
q0m2 $=\|0\rangle$	q1m2 $=\|0\rangle$	$1/\sqrt{2}$	0	$1/\sqrt{2}$	0
	q1m2 $=\|1\rangle$	0	$1/\sqrt{2}$	0	$1/\sqrt{2}$
q0m2 $=\|1\rangle$	q1m2 $=\|0\rangle$	0	$1/\sqrt{2}$	0	$-1/\sqrt{2}$
	q1m2 $=\|1\rangle$	$1/\sqrt{2}$	0	$-1/\sqrt{2}$	0

Tensor network contraction order

- Contraction ordering says the order in which edges are contracted.
- To minimize computation and memory requirements, best to avoid forming large intermediate tensors.

- Akin to the classic dynamic programming problem of optimal chain matrix multiplication.

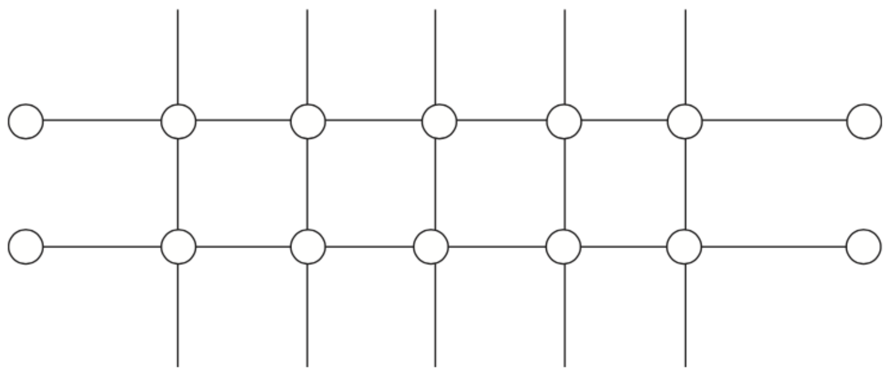


Figure 9.4: Part of a generic tensor network, consisting of ten rank-4 tensors and four rank-1 tensors.

Ding and Chong. Quantum Computer Systems: Research for Noisy Intermediate-Scale Quantum Computers.

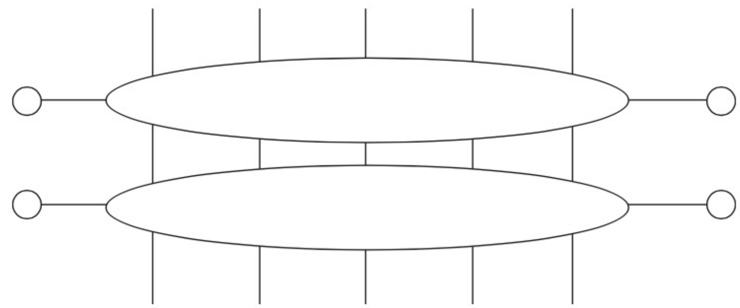


Figure 9.5: First strategy of contraction that results in two rank-12 tensors and four rank-1 tensors. Then contracting the two rank-12 tensors involves contracting 5 edges at once, by summing over 2^5 terms.

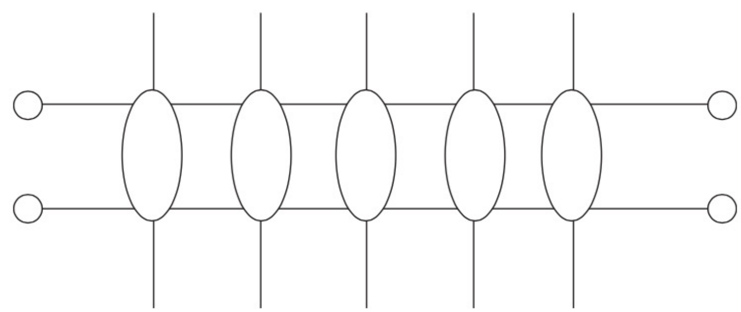


Figure 9.6: Second strategy of contraction that results in five rank-6 tensors and four rank-1 tensors. Then contracting the five rank-6 tensors involves contracting from left to right 2 edges at a time, by summing over 2^2 terms four times.

Ding and Chong. Quantum Computer Systems: Research for Noisy Intermediate-Scale Quantum Computers.

estimate in [7].

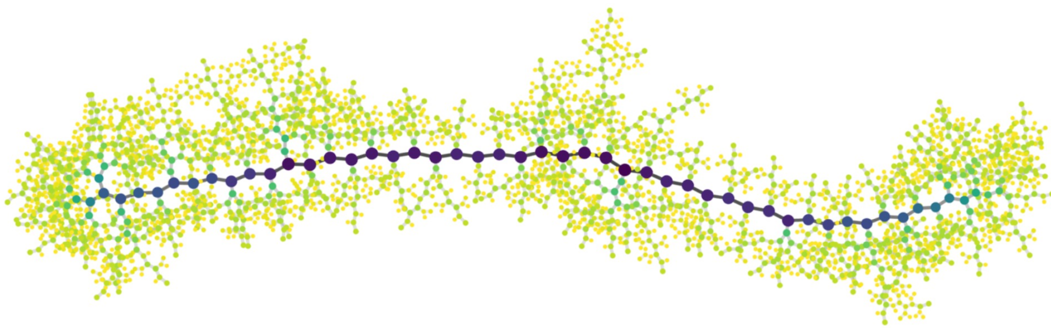


Figure 1: Tensor network contraction of a quantum circuit from the random circuit family [7], visualized as a binary contraction tree. Each node in the tree represents a step in the contraction. Larger, darker nodes indicate more expensive steps. The central stem dominates the overall contraction cost.

Several challenges have been levied against quantum supremacy. The most relevant to this

Cupjin Huang et al., Classical Simulation of Quantum Supremacy Circuits, 2020.

Complexity of quantum circuit simulation via tensor network contraction

- In more detail, cost of simulating the quantum circuit is O(exp(treewidth))
- …where else did we recently see a bound on simulation cost?

Primary Sources

- Markov and Shi. Simulating Quantum Computation by Contracting Tensor Networks. SIAM. 2008.
- Fried, Sawaya, Cao, Kivlichan, Romero, Aspuru-Guzik. qTorch: The quantum tensor contraction handler.
- Biamonte and Bergholm. Quantum Tensor Networks in a Nutshell. 2017.
- Biamonte. Lectures on Quantum Tensor Networks: a pathway to modern diagrammatic reasoning. 2019.
- Kissinger and van de Wetering. Simulating quantum circuits with ZX-calculus reduced stabiliser decompositions. 2021.