Quantum computing fundamentals: one $\wedge$ qubit

or

few

Yipeng Huang

Rutgers University

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Announcements

The class so far

1. Introductions on Canvas discussions. Important for me and classmates to know your interests.
2. Reading: Preskill. "Quantum Computing in the NISQ era and beyond." Describes current state of quantum computing impact and development. Discuss by posting one question and one answer—can be anything.

Intermediate-term class plan

Where we are headed in first month

1. Fundamental rules of quantum computing
2. Basic quantum algorithms
3. Programming examples in Google Cirq
4. A NISQ algorithm: quantum approximate optimization algorithm
5. Programming assignment on QAOA in Cirq

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states

dynamics

composition

measurement

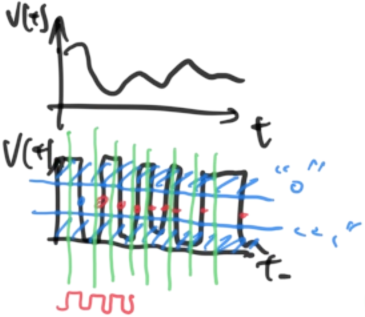
postulates.

state

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The binary abstraction



High, low voltage

Adds resilience against noise.

binary

discrete. time

Representation as a state vector

- $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$ We pronounce this "ket" 0
- $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$ We pronounce this "ket" 1

States

dynamics

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

$0 \leftrightarrow 1$

"not"

$$X|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

The NOT gate

Matrix representation of NOT operator: $X = \sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

- $X\,|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$
- $X\,|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$

Circuit diagram representation:

states

dynamics

$$H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

superposition

$$HH|0\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

States

dynamics

composition

$$|0\rangle \otimes |0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix}1\\0\end{bmatrix} \\ 0\begin{bmatrix}1\\0\end{bmatrix}\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = |00\rangle = |0\rangle$$

$$H|0\rangle \otimes H|1\rangle = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} \otimes \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix} \\ \frac{1}{\sqrt{2}}\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}+1\\-1\\+1\\-1\end{bmatrix} = \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle - \frac{1}{2}|3\rangle$$

state

dynamics

composition

dynamics

$$\text{cnot} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\text{cnot}\,|00\rangle = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{cnot}\,|01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{cnot}\,|10\rangle = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |11\rangle$$

$$\text{cnot}\,|11\rangle = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |10\rangle$$

c t

$$|00\rangle \xrightarrow{\text{cnot}} |00\rangle$$

$$|01\rangle \longrightarrow |01\rangle$$

$$|10\rangle \longrightarrow |11\rangle$$

$$|11\rangle \longrightarrow |10\rangle$$

cnot

controlled-X

The SWAP gate

Matrix representation of SWAP operator: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- $SWAP\,|00\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00\rangle$
- $SWAP\,|10\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle$

Circuit diagram representation:

The CNOT gate

Matrix representation of CNOT operator: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

- $CNOT\,|01\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle$
- $CNOT\,|11\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |10\rangle$

Circuit diagram representation:

The CSWAP gate

Matrix representation of CSWAP operator:

(On the board)

Circuit diagram representation:

The CSWAP gate

Matrix representation of CSWAP operator:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & SWAP \end{array}\right]$$

Circuit diagram representation:

The CCNOT (aka Toffoli) gate

Matrix representation of CCNOT operator:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & CNOT \end{array}\right]$$

Circuit diagram representation:

Universal classical computation

Toffoli (CCNOT) gate can represent all classical computation

(How?)

Universal classical computation

Toffoli (CCNOT) gate can represent all classical computation

1. All Boolean expressions can be phrased as either CNF or DNF.
2. AND, OR, and NOT operations are universal.
3. Either NAND or NOR are individually universal.
4. CCNOT implements NAND. (Feed $|1\rangle$ into target qubit). To see this:
   - Write down truth table for NAND.
   - Write down unitary matrix for CCNOT.
   - Write down truth table for CCNOT.
5. So, CCNOT is universal for classical logic.

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The Hadamard gate

Matrix representation of Hadamard operator: $H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

- $H\left|0\right\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle$
- $H\left|1\right\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{1}{\sqrt{2}}\left|1\right\rangle$

Circuit diagram representation:

Superposition

Single qubit state

- $\alpha |0\rangle + \beta |1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$
- Amplitudes $\alpha, \beta \in \mathbb{C}$
- $|\alpha|^2 + |\beta|^2 = 1$
- The above constraints require that qubit operators are unitary matrices.

Many physical phenomena can be in superposition and encode qubits

- Polarization of light in different directions
- Electron spins (Intel solid state qubits)
- Atom energy states (UMD, IonQ ion trap qubits)
- Quantized voltage and current (IBM, Google superconducting qubits)

If multiple discrete values are possible (e.g., atom energy states, voltage and current), we pick (bottom) two for the binary abstraction.

Interference

Amplitudes can positively and negatively interfere

- $HH\,|0\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$
- $HH\,|1\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2} \\ \frac{1}{2} + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$

Circuit diagram representation:

Measurement

These rules, states, and operators model real quantum phenomena

- States in our examples cannot be merely classical or probabilistic

Double slit experiment

`https://www.youtube.com/watch?v=Q1YqgPAtzho`

Textbook formalism

For an introductory textbook on the quantum computing formalism, I recommend:

`https://www.lassp.cornell.edu/mermin/qcomp/CS483.html`

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