Quantum computing fundamentals: State, Composition, Dynamics

Yipeng Huang

Rutgers University

January 19, 2024

Table of contents

state
composition
dynamics

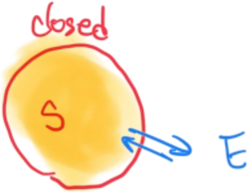
Postulates of quantum mechanics

The state of a single qubit
- Superposition
- Bloch sphere

The state of multiple qubits
- Tensor product
- Entanglement
- No-cloning theorem

The evolution of qubit states

Postulates of quantum mechanics



1. State space
2. Composite systems
3. Evolution
4. Quantum measurement

1, 2, and 3 are linear and describe closed quantum systems. 4 is nonlinear and describes open quantum systems.

Table of contents

Postulates of quantum mechanics

The state of a single qubit
- Superposition
- Bloch sphere

The state of multiple qubits
- Tensor product
- Entanglement
- No-cloning theorem

The evolution of qubit states

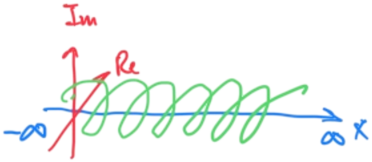
Quantum postulate 1: State space

The position or momentum of a physical system is described as a wavefunction

general

- Assuming continuous state space:

$$|\psi\rangle = \int_{-\infty}^{\infty} \psi(x)\,|x\rangle\, dx$$



$$|x\rangle$$

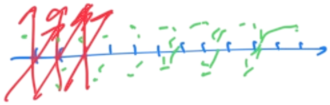
are orthonormal

$$\psi(x) \in \mathbb{C}$$

- Assuming discrete state space:

specialization

$$|\psi\rangle = \sum_{i=0}^{\infty} \psi(x_i)\,|x_i\rangle$$



$$\psi(x) \in \mathbb{C}$$

Quantum postulate 1: State space

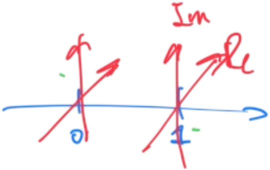
The position or momentum of a physical system is described as a wavefunction

- Assuming discrete state space:

$$|\psi\rangle = \sum_{i=0}^{\infty} \psi(x_i)\,|x_i\rangle$$

$$\psi(x) \in \mathbb{C}$$

- Assuming discrete binarized state space:



$$|\psi\rangle = \sum_{i=0}^{1} \psi(x_i)\,|x_i\rangle$$

$$\psi(x) \in \mathbb{C}$$

The Hadamard gate

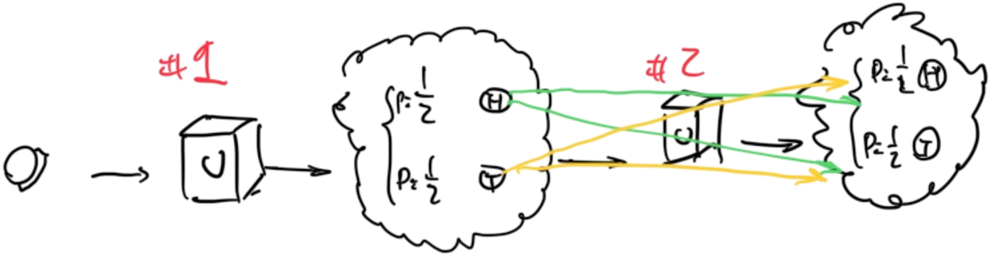
$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$\alpha, \beta \in \mathbb{C}$

Matrix representation of Hadamard operator: $H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

- $H|0\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
- $H|1\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Circuit diagram representation: $|0\rangle \text{ — } \boxed{H} \text{ — } \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$



classical probability

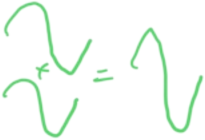
Interference

superposition

Amplitudes can positively and negatively interfere

- $HH\left|0\right\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left|0\right\rangle$
- $HH\left|1\right\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2} \\ \frac{1}{2} + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left|1\right\rangle$

offset phase



Circuit diagram representation:

$$\left|0\right\rangle - \underbrace{\boxed{H} - \boxed{H}} - \left|0\right\rangle = \left|0\right\rangle - \boxed{I} - \left|0\right\rangle$$

Superposition

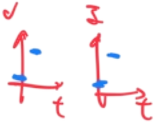
Single qubit state

- $\alpha |0\rangle + \beta |1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$
- Amplitudes $\alpha, \beta \in \mathbb{C}$
- $|\alpha|^2 + |\beta|^2 = 1$
- The above constraints require that qubit operators are unitary matrices.

Many physical phenomena can be in superposition and encode qubits

- Polarization of light in different directions
- Electron spins (Intel solid state qubits) e^-
- Atom energy states (UMD, IonQ ion trap qubits)

- Quantized voltage and current (IBM, Google superconducting qubits)



If multiple discrete values are possible (e.g., atom energy states, voltage and current), we pick (bottom) two for the binary abstraction.

Bloch sphere

Representation of pure states of a single qubit

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

$\theta = 0$ $\quad |\psi\rangle = \cos 0|0\rangle + \sin 0|1\rangle = |0\rangle$

- θ polar angle
- ϕ azimuthal angle

$\theta = \pi$ $\quad |\psi\rangle = \cos\frac{\pi}{2}|0\rangle + \sin\frac{\pi}{2}|1\rangle = 1|1\rangle$

Euler's formula

$$e^{i\phi} = \cos\phi + i\sin\phi$$

$\theta = \pi, \ \phi = \frac{\pi}{2}$ $\quad |\psi\rangle = \cos\frac{\pi}{2}|0\rangle + e^{i\frac{\pi}{2}}\sin\frac{\pi}{2}|1\rangle = i|1\rangle$

Global Phase X

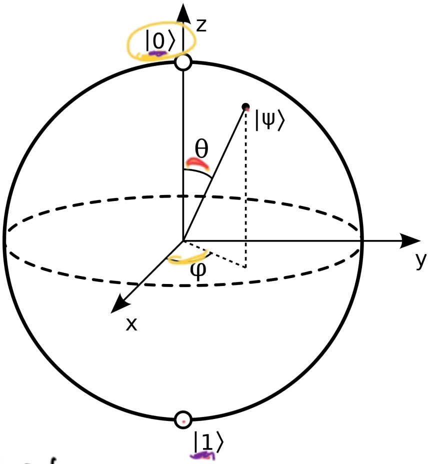


Figure: Source: Wikimedia

Bloch sphere

~~$|q_1\rangle$ two qubit~~

Representation of pure states of a single qubit

$$|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle$$

- θ polar angle
- ϕ azimuthal angle

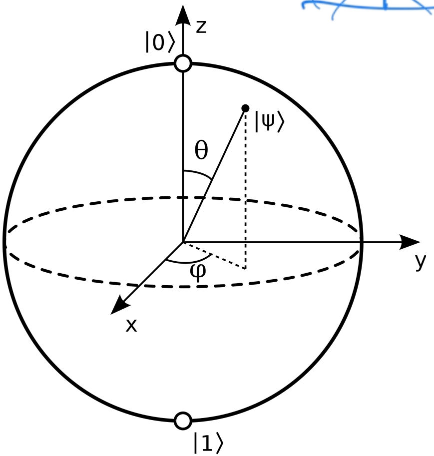


Figure: Source: Wikimedia

Euler's formula

$$e^{i\phi} = \cos\phi + i\sin\phi$$

Bloch sphere

$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|0\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$

$\theta = \frac{\pi}{2}, \ \phi = \emptyset : \cos\frac{\pi}{4}|0\rangle + e^{i\phi}\sin\frac{\pi}{4}|1\rangle$

$= \frac{1}{\sqrt{2}}|0\rangle + e^{i\phi}\frac{1}{\sqrt{2}}|1\rangle$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

$$\Rightarrow e^{i\phi} = \cos\phi + i\sin\phi$$

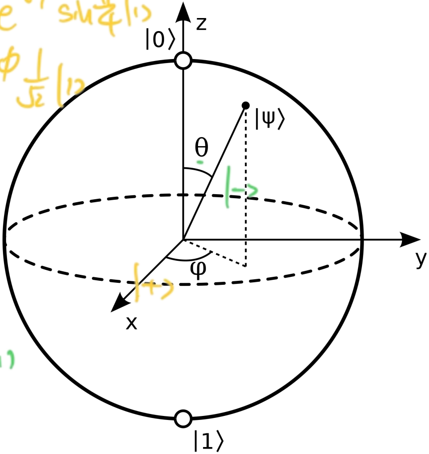


Figure: Source: Wikimedia

Important locations on the Bloch sphere

- $|+\rangle = H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
- $|-\rangle = H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

$\theta = \frac{\pi}{2}, \ \phi =$

$e^{i\phi} = -1 = \cos\phi + i\sin\phi$

$\phi = \pi$

Bloch sphere

$$|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle$$

$$e^{i\phi} = \cos\phi + i\sin\phi$$

$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ "not"

$$R_x(\pi) = \cos\frac{\pi}{2} I - i\sin\frac{\pi}{2} X = \cancel{0} - i\,1\,X = -iX$$

Rotations around the Bloch sphere

- $$R_x(\theta) = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} X$$
- $$R_y(\theta) = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Y$$
- $$R_z(\theta) = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Z$$

$$R_x(\pi)|0\rangle = -iX|0\rangle = -i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -i\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -i|1\rangle \equiv |1\rangle$$

global phase.

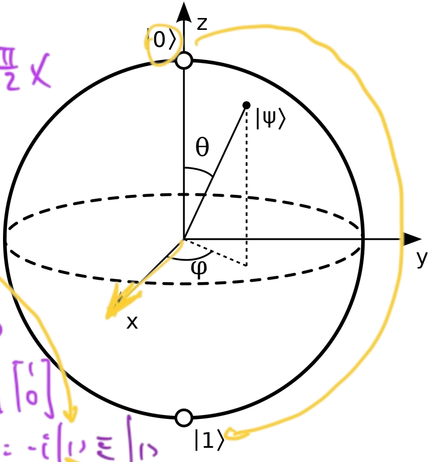


Figure: Source: Wikimedia

Bloch sphere

$$|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle$$

$$e^{i\phi} = \cos\phi + i\sin\phi$$

Rotations around the Bloch sphere

- $$R_x(\theta) = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}X$$
- $$R_y(\theta) = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Y$$
- $$R_z(\theta) = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Z$$

error correction

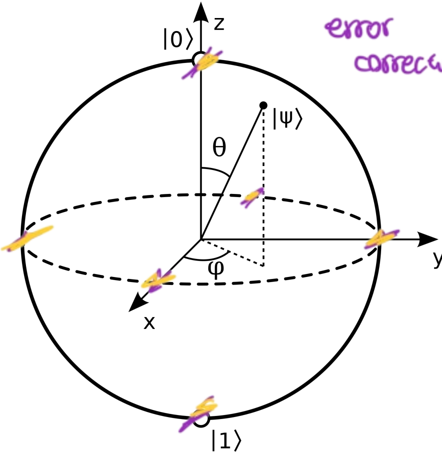


Figure: Source: Wikimedia

Bloch sphere

$$|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle$$

$$e^{i\phi} = \cos\phi + i\sin\phi$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$R_z(\pi)|+\rangle = \left(\cos\frac{\pi}{2} I - i\sin\frac{\pi}{2} Z\right)\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = -i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = -i\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = -i|-\rangle$$

global phase

Rotations around the Bloch sphere

- $$R_x(\theta) = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} X$$
- $$R_y(\theta) = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Y$$
- $$R_z(\theta) = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Z$$

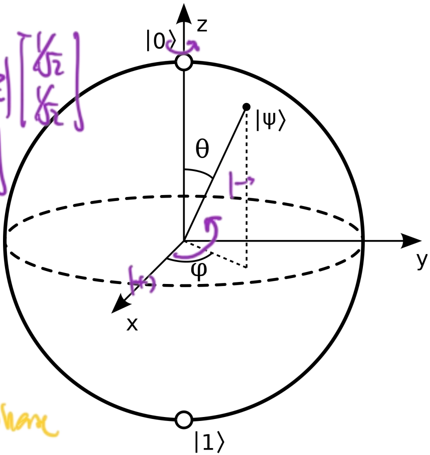


Figure: Source: Wikimedia

Table of contents

Postulates of quantum mechanics

The state of a single qubit
- Superposition
- Bloch sphere

The state of multiple qubits
- Tensor product
- Entanglement
- No-cloning theorem

The evolution of qubit states

Quantum postulate 2: Composite systems

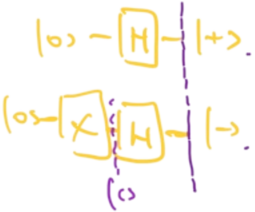
multiple qubits

The state space of $\underbrace{\text{composite systems}}$ is the tensor product of state space of component systems.

Multiple qubits: the tensor product

Tensor product (also known as Kronecker product) of state vectors

$$|+\rangle \otimes |-\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \\ \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2} \\ \frac{1}{2} \\ \frac{-1}{2} \end{bmatrix} = \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$



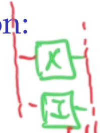
Multiple qubits: the tensor product

Tensor product of unitary matrices

$$X \otimes I \left(\frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{2}} |11\rangle \right) = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} =$$

$$\begin{bmatrix} 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{2}} |11\rangle$$

Circuit diagram representation:



$\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$

$\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$

Multiple qubits: the tensor product

Tensor product of state vectors

$$X\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes I|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Circuit diagram representation:

Multiple qubits: the tensor product

Recursive routine → base case - → recursive step-

proof by induction → base case ↘ induction step.

Exercise: proof by induction about the Hadamard transform

Show that $|+\rangle^{\otimes n} = \frac{1}{2^{n/2}} \sum_{m=0}^{2^n-1} |m\rangle$

$$\underbrace{|+\rangle \otimes |+\rangle \otimes |+\rangle}_{n=3} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \otimes \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \frac{1}{2^{3/2}} \sum_{m=0}^{2^3-1} |m\rangle$$

$$= \frac{1}{2\sqrt{2}} \left(|0\rangle + |1\rangle + |2\rangle \cdots + |7\rangle \right)$$

-H-
-H-
-H-

reading assignment

introduction

pset #1 soon...

recommend: QC for non phy :-

Entangled states: Bell state circuit

Bell state circuit

$$|00\rangle \xrightarrow{H \otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |\Phi^+\rangle$$

Can $|\Phi^+\rangle$ be treated as the tensor product (composition) of two individual qubits?

Prove that the Bell state cannot be factored into two single-qubit states

Bell state circuit

$$|00\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = |\Phi^+\rangle$$

Can $|\Phi^+\rangle$ be treated as the tensor product (composition) of two individual qubits?

No.

Bell states form an orthogonal basis set

1. $|00\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) = |\Phi^+\rangle$
2. $|01\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|01\rangle + |11\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|01\rangle + |10\rangle \Big) = |\Psi^+\rangle$
3. $|10\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle - |10\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle - |11\rangle \Big) = |\Phi^-\rangle$
4. $|11\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|01\rangle - |11\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|01\rangle - |10\rangle \Big) = |\Psi^-\rangle$

No-cloning theorem

There is no way to duplicate an arbitrary quantum state

Suppose a cloning operation U_c exists. Then:

- $$U_c(|\phi\rangle \otimes |\omega\rangle) = |\phi\rangle \otimes |\phi\rangle \,,$$
  $$U_c(|\psi\rangle \otimes |\omega\rangle) = |\psi\rangle \otimes |\psi\rangle \,,$$
  for arbitrary states $|\phi\rangle\,, |\psi\rangle$ we wish to copy.
- The overlap of the initial states is:

$$\langle\phi| \otimes \langle\omega|\,|\psi\rangle \otimes |\omega\rangle = \langle\phi|\,|\psi\rangle \cdot \langle\omega|\,|\omega\rangle = \langle\phi|\,|\psi\rangle$$

No-cloning theorem

There is no way to duplicate an arbitrary quantum state

Suppose a cloning operation U_c exists. Then:

- $$U_c(|\phi\rangle \otimes |\omega\rangle) = |\phi\rangle \otimes |\phi\rangle \,,$$
  $$U_c(|\psi\rangle \otimes |\omega\rangle) = |\psi\rangle \otimes |\psi\rangle \,,$$
  for arbitrary states $|\phi\rangle\,, |\psi\rangle$ we wish to copy.
- The overlap of the final states is:
  $$\langle\phi| \otimes \langle\phi| \, |\psi\rangle \otimes |\psi\rangle = \langle\phi| \, |\psi\rangle \cdot \langle\phi| \, |\psi\rangle = (\langle\phi| \, |\psi\rangle)^2$$
- The overlap of the final states is also:
  $$\langle\phi| \otimes \langle\phi| \, |\psi\rangle \otimes |\psi\rangle = \langle\phi| \otimes \langle\omega| \, U^\dagger U \, |\psi\rangle \otimes |\omega\rangle = \langle\phi| \otimes \langle\omega| \, |\psi\rangle \otimes |\omega\rangle = \langle\phi| \, |\psi\rangle$$
- $(\langle\phi| \, |\psi\rangle)^2 = \langle\phi| \, |\psi\rangle$, so $\langle\phi| \, |\psi\rangle = 0$, or $\langle\phi| \, |\psi\rangle = 1$, $|\phi\rangle$ and $|\psi\rangle$ cannot be arbitrary states as claimed.

Table of contents

Postulates of quantum mechanics

The state of a single qubit
- Superposition
- Bloch sphere

The state of multiple qubits
- Tensor product
- Entanglement
- No-cloning theorem

The evolution of qubit states

Quantum postulate 3: Evolution

The time evolution of a state follows the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

- Comes from the conservation of total energy in the closed system, one of the observables from the system state.
- Itself reflects a time-invariance.

$$\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{-iH}{\hbar}|\psi(t)\rangle$$

$$|\psi(t)\rangle = e^{\frac{-iH}{\hbar}}|\psi(t)\rangle$$

Quantum postulate 3: Evolution

The evolution of a closed quantum system is a unitary transformation.

$$|\psi(t = t_1)\rangle = U\,|\psi(t = t_0)\rangle$$

- $|\psi_1\rangle = U\,|\psi_0\rangle$
- In a closed quantum system, $\langle\psi_1|\,|\psi_1\rangle = \langle\psi_0|\,U^\dagger U\,|\psi_0\rangle = \langle\psi_0|\,|\psi_0\rangle = 1$
- $U^\dagger U = I$, $U^\dagger = U^{-1}$; Such matrices U are unitary

Quantum postulate 3: Evolution

From unitary transformations we can show Hamiltonians in closed quantum systems must be hermitian

- $U|\psi\rangle = e^{\frac{-iH}{\hbar}}|\psi\rangle$
- $U^{\dagger}|\psi\rangle = e^{\frac{-(iH)^{\dagger}}{\hbar}}|\psi\rangle$
- $U^{\dagger}|\psi\rangle = U^{-1}|\psi\rangle = e^{\frac{iH}{\hbar}}|\psi\rangle$
- $(iH)^{\dagger} = -iH$, $A = iH$; such matrices A are called anti-Hermitian a.k.a. skew-Hermitian
- If iH is skew-Hermitian, H is Hermitian a.k.a. self-adjoint: $H^{\dagger} = H$