# Quantum computing fundamentals: Dynamics, Measurement

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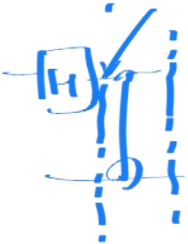
state
composition ; entangled
dynamics
measurement

# Postulates of quantum mechanics




1. State space
2. Composite systems
3. Evolution
4. Quantum measurement

QC

1, 2, and 3 are linear and describe closed quantum systems. 4 is nonlinear and describes open quantum systems.

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# No-cloning theorem

There is no way to duplicate an arbitrary quantum state

Suppose a cloning operation $U_c$ exists. Then:

- $$U_c(|\phi\rangle \otimes |\omega\rangle) = |\phi\rangle \otimes |\phi\rangle\,,$$
  $$U_c(|\psi\rangle \otimes |\omega\rangle) = |\psi\rangle \otimes |\psi\rangle\,,$$
  for arbitrary states $|\phi\rangle$ , $|\psi\rangle$ we wish to copy.
- The overlap of the initial states is:

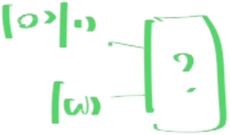
$$\langle\phi| \otimes \langle\omega|\,|\psi\rangle \otimes |\omega\rangle = \langle\phi|\,|\psi\rangle \cdot \langle\omega|\,|\omega\rangle = \langle\phi|\,|\psi\rangle$$

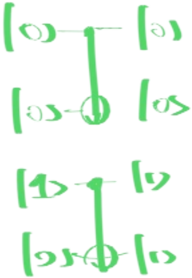
# No-cloning theorem

## There is no way to duplicate an arbitrary quantum state

Suppose a cloning operation $U_c$ exists. Then:

- $$U_c(|\phi\rangle \otimes |\omega\rangle) = |\phi\rangle \otimes |\phi\rangle\,,$$
  $$U_c(|\psi\rangle \otimes |\omega\rangle) = |\psi\rangle \otimes |\psi\rangle\,,$$
  for arbitrary states $|\phi\rangle$, $|\psi\rangle$ we wish to copy.
- The overlap of the final states is:
  $$\langle\phi| \otimes \langle\phi|\, |\psi\rangle \otimes |\psi\rangle = \langle\phi|\,|\psi\rangle \cdot \langle\phi|\,|\psi\rangle = (\langle\phi|\,|\psi\rangle)^2$$
  $U_c^\dagger U_c = I$
- The overlap of the final states is also:
  $$\langle\phi| \otimes \langle\phi|\, |\psi\rangle \otimes |\psi\rangle = \langle\phi| \otimes \langle\omega|\, U^\dagger U\, |\psi\rangle \otimes |\omega\rangle = \langle\phi| \otimes \langle\omega|\, |\psi\rangle \otimes |\omega\rangle = \langle\phi|\,|\psi\rangle$$
- $(\langle\phi|\,|\psi\rangle)^2 = \langle\phi|\,|\psi\rangle$, so $\langle\phi|\,|\psi\rangle = 0$, or $\langle\phi|\,|\psi\rangle = 1$, $|\phi\rangle$ and $|\psi\rangle$ cannot be arbitrary states as claimed.



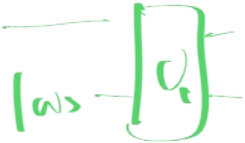

$|w\rangle$

?

$|w\rangle = |0\rangle$

? = cnot

$|+\rangle, |-\rangle$



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# Quantum postulate 3: Evolution

$U|\psi\rangle$

## The time evolution of a state follows the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle$$

- Comes from the conservation of total energy in the closed system, one of the observables from the system state.
- Itself reflects a time-invariance.

$$\frac{\partial}{\partial t} |\psi(t)\rangle = \frac{-iH}{\hbar} |\psi(t)\rangle$$

$$|\psi(t)\rangle = e^{\frac{-iH}{\hbar}} |\psi(t)\rangle$$

$U$

# Quantum postulate 3: Evolution

The evolution of a closed quantum system is a unitary transformation.

$$|\psi(t = t_1)\rangle = U\,|\psi(t = t_0)\rangle$$

- $|\psi_1\rangle = U\,|\psi_0\rangle$
- In a closed quantum system, $\langle\psi_1|\,|\psi_1\rangle = \langle\psi_0|\,U^\dagger U\,|\psi_0\rangle = \langle\psi_0|\,|\psi_0\rangle = 1$
- $U^\dagger U = I$, $U^\dagger = U^{-1}$; Such matrices $U$ are unitary

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$Y^\dagger = Y^{T*} = \left(\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\right)^* = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$U^\dagger = U^{-1}$$

$$Y \cdot Y^\dagger = I$$

H

I

X

Y

Z

cnot

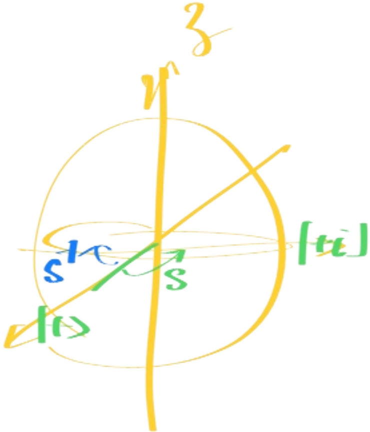
$$cy = \left[\begin{array}{c|c} I & \\ \hline & Y \end{array}\right]$$

$S = \sqrt{Z} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

$S^2 = Z$

$S|+\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix} = |+i\rangle$




$S^{\dagger} = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$

$S \cdot S^{\dagger} = I \quad \therefore \quad S^{-1} = S^{\dagger}$

# Quantum postulate 3: Evolution

From unitary transformations we can show Hamiltonians in closed quantum systems must be hermitian

- $U\left|\psi\right\rangle = e^{\frac{-iH}{\hbar}}\left|\psi\right\rangle$
- $U^{\dagger}\left|\psi\right\rangle = e^{\frac{-(iH)^{\dagger}}{\hbar}}\left|\psi\right\rangle$
- $U^{\dagger}\left|\psi\right\rangle = U^{-1}\left|\psi\right\rangle = e^{\frac{iH}{\hbar}}\left|\psi\right\rangle$
- $(iH)^{\dagger} = -iH$, $A = iH$; such matrices A are called anti-Hermitian a.k.a. skew-Hermitian
- If $iH$ is skew-Hermitian, $H$ is Hermitian a.k.a. self-adjoint: $H^{\dagger} = H$

$A = iH$

$H$

$U$

$U^{\dagger} = U^{-1}$

$U = U^{\dagger}$

$U \cdot U |\varphi\rangle$

$= U U^{\dagger} |\varphi\rangle$

$= U U^{-1} |\varphi\rangle$

$= I |\varphi\rangle$

# Universal classical computation

Toffoli (CCNOT) gate can represent all classical computation

(How?)

# Functional completeness

$(A \land B) \lor (A \land \lnot B)$

All computation on binary variables can be represented as

$$f(x) = y$$

$$x \in \{0,1\}^n; y \in \{0,1\}^m$$

All Boolean expressions can be phrased as either CNF (and of ors) or DNF (or of ands).

Various sets of logic gates are functionally complete

- {NOT,AND,OR}
- {NAND}
- {NOR}

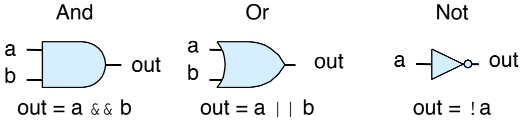
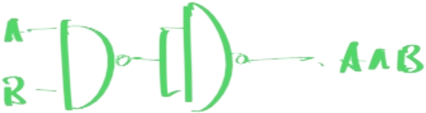



Figure: Source: CS:APP

nand is complete

not as nand


and as nand




or as nand

# Reversible operations

| A | B | A nand B |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

## Toffoli (CCNOT) gate can represent all classical computation

CCNOT implements NAND

- Write down truth table for NAND.
- Write down truth table for CCNOT.
- Feed $|1\rangle$ into target qubit.

Creating classical computers out of purely reversible logic is a way to push the extremes of computing energy efficiency.

$$
CCNOT = \begin{pmatrix}
1 & & & & & & & \\
 & 1 & & & & & & \\
 & & 1 & & & & & \\
 & & & 1 & & & & \\
 & & & & 1 & & & \\
 & & & & & 1 & & \\
 & & & & & & & 1 \\
 & & & & & & 1 & 
\end{pmatrix}
$$

Logic

000 → =

001 → =

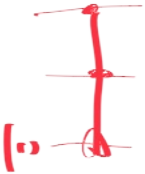
010 → =

011 → =

100 → =

101 → =

$\left\{\begin{array}{l} 110 \to 111 \\ 111 \to 110 \end{array}\right.$



$$CCNOT = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & & 1 \\ & & & & & & 1 & \end{pmatrix}$$

Logic

~~000 →~~

001 →

~~010 →~~

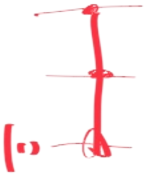
011 →

~~100 →~~

101 →

~~110 → 111~~

111 → 110



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$|0\rangle, |1\rangle \rightarrow 0, 1$

# Quantum postulate 4: Measurement

$$p(|0\rangle) = \langle + | \left( |0\rangle\langle 0| \right) |+\rangle = \langle + | \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} |+\rangle = \frac{1}{2}$$

When a closed quantum system with state $|\psi\rangle$ interacts with the environment, measurement takes place:

- The probability of the post-measurement state being in state $|a_n\rangle$ is: $p(|a_n\rangle) = \langle\psi| \, |a_n\rangle \langle a_n| \, |\psi\rangle = |\langle a_n| \, |\psi\rangle|^2$
- The state of the quantum system is then renormalized to $\frac{|a_n\rangle\langle a_n||\psi\rangle}{\sqrt{p(|a_n\rangle)}}$

Let's practice this with measuring a single-qubit state.

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$P(|10\rangle) = \langle\Phi^+|\,|10\rangle\langle 10|\,|\Phi^+\rangle$$

$$= \langle\Phi^+|\begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \end{bmatrix}|\Phi^+\rangle = \left|\langle 10|\,|\Phi^+\rangle\right|^2 = 0^2 = \phi$$

$$P(|11\rangle) = \langle\Phi^+|\,|11\rangle\langle 11|\,|\Phi^+\rangle = \frac{1}{2}$$


$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$\begin{cases} P = \frac{1}{2} & 0,0 \\ P = \frac{1}{2} & 1,1 \end{cases}$$

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Algorithms

protocols

↳ superdense coding

↳ teleportation

algorithms.

↳

# Entangled states: Bell state circuit

### Bell state circuit

$$|00\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big( |00\rangle + |10\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big( |00\rangle + |11\rangle \Big) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |\Phi^+\rangle$$

Can $|\Phi^+\rangle$ be treated as the tensor product (composition) of two individual qubits?

# Prove that the Bell state cannot be factored into two single-qubit states

Bell state circuit

$$|00\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = |\Phi^+\rangle$$

Can $|\Phi^+\rangle$ be treated as the tensor product (composition) of two individual qubits?

No.

# Bell states form an orthogonal basis set

1. $|00\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big( |00\rangle + |10\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big( |00\rangle + |11\rangle \Big) = |\Phi^+\rangle$
2. $|01\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big( |01\rangle + |11\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big( |01\rangle + |10\rangle \Big) = |\Psi^+\rangle$
3. $|10\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big( |00\rangle - |10\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big( |00\rangle - |11\rangle \Big) = |\Phi^-\rangle$
4. $|11\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big( |01\rangle - |11\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big( |01\rangle - |10\rangle \Big) = |\Psi^-\rangle$

# Superdense coding

## Transmit 2 bits of classical information by sending 1 qubit

1. Alice wishes to tell Bob two bits of information: 00, 01, 10, or 11.
2. Alice and Bob each have one qubit of a Bell pair in state $|P\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}}\Big(\,|00\rangle + |11\rangle\,\Big)$.
3. Alice performs $I$, $X$, $Z$, or $ZX$ on her qubit; she then sends her qubit to Bob.
4. Bob measures in the Bell basis to receive 00, 01, 10, or 11.

## Superdense coding circuit

`https://github.com/quantumlib/Cirq/blob/master/examples/superdense_coding.py`

# Superdense coding

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3. Alice performs $I$, $X$, $Z$, or $ZX$ on her qubit; she then sends her qubit to Bob.
4. Bob measures in the Bell basis to receive 00, 01, 10, or 11.

## Alice applies different operators on her qubit so Bob measures the message

1. $|P\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) \xrightarrow{I\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle\Big) \xrightarrow{H\otimes I} |00\rangle$
2. $|P\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) \xrightarrow{X\otimes I} \frac{1}{\sqrt{2}}\Big(|10\rangle + |01\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|11\rangle + |01\rangle\Big) \xrightarrow{H\otimes I} |01\rangle$
3. $|P\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) \xrightarrow{Z\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle - |11\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle - |10\rangle\Big) \xrightarrow{H\otimes I} |10\rangle$
4. $|P\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) \xrightarrow{ZX\otimes I} \frac{1}{\sqrt{2}}\Big(-|10\rangle + |01\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(-|11\rangle + |01\rangle\Big) \xrightarrow{H\otimes I} |11\rangle$