Basic quantum algorithms: Deutsch / Deutsch-Jozsa

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XZ

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$iXZ = Y$

Quaternions

Pauli matrices.

$\mathbb{Z}$

$\mathbb{R}$,

$\mathbb{C}$ $i^2 = -1$

$\mathbb{H}$, $i^2 = j^2 = k^2 = ijk$,

graphics

robotics

Promise algorithms vs. unstructured search

Quantum algorithms offer exponential speedup in “promise” problems

A progression of related algorithms:

1. Deutsch’s
2. Deutsch-Jozsa
3. Bernstein-Vazirani
4. Simon’s
5. Shor’s

1992 (Deutsch’s) — 1996 (Shor’s)

PA1

a promise about a function to be exploited

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n=1

n>1

Deutsch-Jozsa algorithm: simplest quantum algorithm showing advantage vs. classical

A Heist

- You break into a bank vault. The bank vault has 2^n bars. Three possibilities: all are gold, half are gold and half are fake, or all are fake.
- Even if you steal just one gold bar, it is enough to fund your escape from the country, forever evading law enforcement.
- You do not want to risk stealing from a bank vault with only fake bars.
- You have access to an oracle $f(x)$ that tells you if gold bar x is real.
- Using the oracle sounds the alarm, so you only get to use it once.

Q is for Quantum.

Deutsch-Jozsa algorithm: simplest quantum algorithm showing advantage vs. classical

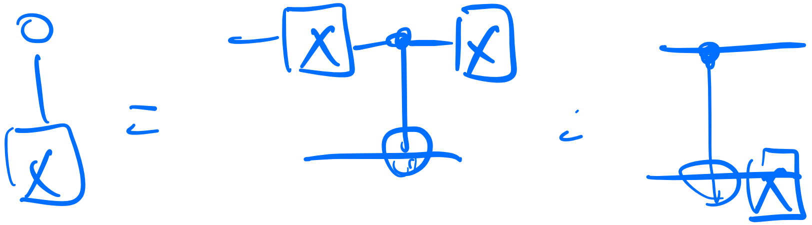
More formal description

- The 2^n bars are either fake or gold. $f : \{0,1\}^n \to \{0,1\}$.
- Three possibilities:
  1. All are fake. f is constant. $f(x) = 0$ for all $x \in \{0,1\}^n$.
  2. All are gold. f is constant. $f(x) = 1$ for all $x \in \{0,1\}^n$.
  3. Half fake half gold. f is balanced.

$$\left|\{x \in \{0,1\}^n : f(x) = 0\}\right| = \left|\{x \in \{0,1\}^n : f(x) = 1\}\right| = 2^{n-1}$$

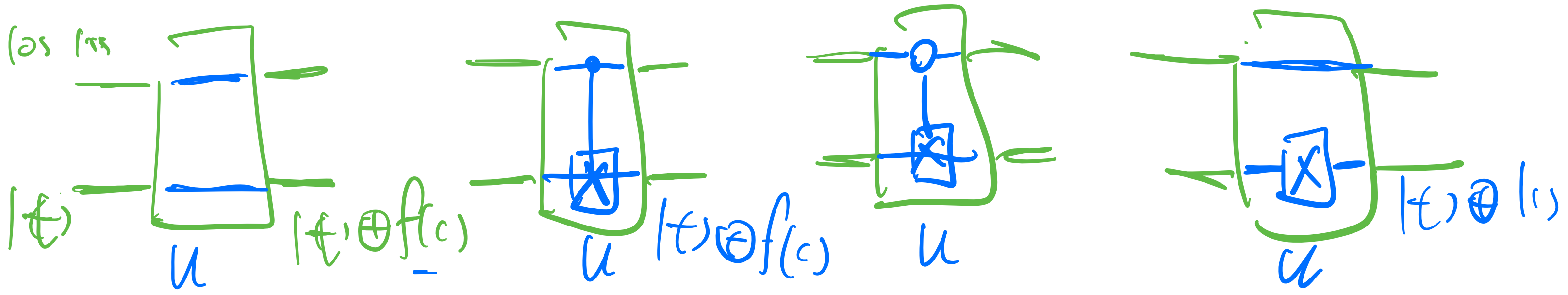
- The oracle U works as follows: $U|c\rangle|t\rangle = |c\rangle|t \oplus f(c)\rangle$
- Try deciding if f is constant or balanced using oracle U only once.

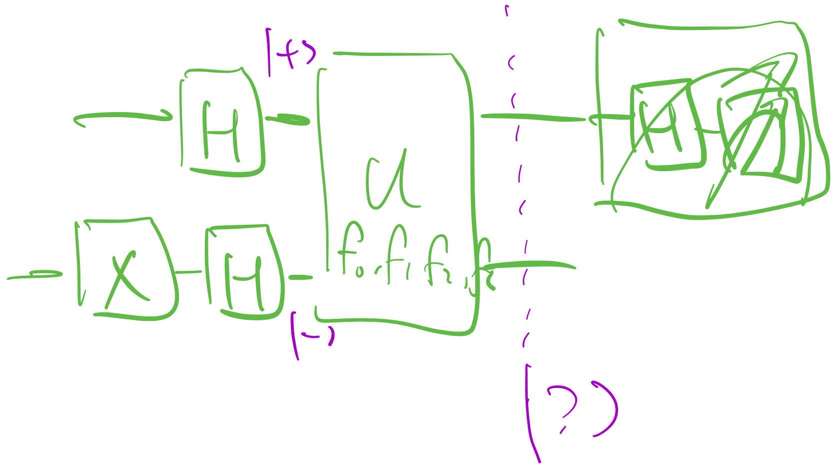
What is in the oracle



For $n = 1$, four possibilities

	f_0	f_1	f_2	f_3
f(0)	0	0	1	1
f(1)	0	1	0	1
	f is constant 0	f is balanced	f is balanced	f is constant 1





$$f_0: \quad U_{f_0}\frac{1}{2}\begin{bmatrix}+1\\-1\\+1\\-1\end{bmatrix} = I\otimes I\,\frac{1}{2}\begin{bmatrix}+1\\-1\\+1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}+1\\-1\\+1\\-1\end{bmatrix} = |+\rangle\otimes|-\rangle$$

$$f_2: \quad U_{f_2}\frac{1}{2}\begin{bmatrix}+1\\-1\\+1\\-1\end{bmatrix} = \left[\begin{array}{cc|cc}0&1&0&0\\1&0&0&0\\\hline 0&0&1&0\\0&0&0&1\end{array}\right]\frac{1}{2}\begin{bmatrix}+1\\-1\\+1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1\\+1\\+1\\-1\end{bmatrix} = -|-\rangle\otimes|-\rangle$$

$$f_3: \quad U_{f_3}\frac{1}{2}\begin{bmatrix}+1\\-1\\+1\\-1\end{bmatrix} = (I\otimes X)\frac{1}{2}\begin{bmatrix}+1\\-1\\+1\\-1\end{bmatrix} = \left[\begin{array}{cc|cc}0&1&0&0\\1&0&0&0\\\hline 0&0&0&1\\0&0&1&0\end{array}\right]\frac{1}{2}\begin{bmatrix}+1\\-1\\+1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1\\+1\\-1\\+1\end{bmatrix}$$

$$= -|+\rangle\otimes|-\rangle$$

CNOT :

$$|00\rangle \to |00\rangle$$
$$|01\rangle \to |01\rangle$$
$$|10\rangle \to |11\rangle$$
$$|11\rangle \to |10\rangle$$

$$\left(\frac{I+X}{2}\right) \otimes \left(\frac{I-X}{2}\right)$$

$$(I \otimes X)\,\text{CNOT} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{CNOT}\ II\ \text{CNOT} = II$$

$$\text{CNOT}\ IX\ \text{CNOT} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = IX$$

CNOT XI CNOT

$$\begin{pmatrix} 1&&& \\ &1&& \\ &&&1 \\ &&1& \end{pmatrix}\begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix}\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix}$$

$$= \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix}\begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix} = XX$$

$$\left(\frac{I+X}{2}\right) \otimes \left(\frac{I-X}{2}\right) \xrightarrow{\text{CNOT}} \left(\frac{I-X}{2}\right) \otimes \left(\frac{I-X}{2}\right)$$

$$\Rightarrow \frac{II - IX + XI - XX}{4} \xrightarrow{\text{CNOT}} \frac{II - IX - XI + XX}{4}$$

CNOT:

$II \to II$

$IX \to IX$

$XI \to XX$

$XX \to XI$

Demonstration of Deutsch-Jozsa for the $n = 1$ case

Output of circuit is $c = 0$ iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle \otimes |1\rangle = |0\rangle |1\rangle = |01\rangle$
2. After first set of Hadamards: $H \otimes H \left(|0\rangle \otimes |1\rangle \right) = H|0\rangle \otimes H|1\rangle = |+\rangle \otimes |-\rangle =$

$$\left(\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \right) \otimes \left(\frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \right) = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

From here, let's take an aside via matrix-vector multiplication to build intuition with interference and phase kickback.

Demonstration of Deutsch-Jozsa for the $n = 1$ case

Output of circuit is $c = 0$ iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle \otimes |1\rangle = |0\rangle\,|1\rangle = |01\rangle$
2. After first set of Hadamards: $H \otimes H\Big(\,|0\rangle \otimes |1\rangle\,\Big) = |+\rangle\,|-\rangle =$
   $$\Big(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\Big)\Big(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\Big) = \tfrac{1}{2}\Big(\,|0\rangle\,(\,|0\rangle - |1\rangle\,) + |1\rangle\,(\,|0\rangle - |1\rangle\,)\Big)$$
3. After applying oracle U:
   $$U\tfrac{1}{2}\Big(\,|0\rangle\,(\,|0\rangle - |1\rangle\,) + |1\rangle\,(\,|0\rangle - |1\rangle\,)\Big) = \tfrac{1}{2}\Big(\,|0\rangle\,(\,|f(0) \oplus 0\rangle - |f(0) \oplus 1\rangle\,) +$$
   $$|1\rangle\,(\,|f(1) \oplus 0\rangle - |f(1) \oplus 1\rangle\,)\Big) = \tfrac{1}{2}\Big(\,|0\rangle\,(\,|f(0)\rangle - |f(\bar{0})\rangle\,) + |1\rangle\,(\,|f(1)\rangle - |f(\bar{1})\rangle\,)\Big)$$

Demonstration of Deutsch-Jozsa for the $n = 1$ case

Output of circuit is $c = 0$ iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle \otimes |1\rangle = |0\rangle |1\rangle = |01\rangle$
2. After first set of Hadamards: $\frac{1}{2}\Big(|0\rangle (|0\rangle - |1\rangle) + |1\rangle (|0\rangle - |1\rangle)\Big)$
3. After applying oracle U: $U\frac{1}{2}\Big(|0\rangle (|0\rangle - |1\rangle) + |1\rangle (|0\rangle - |1\rangle)\Big) =$
   $\frac{1}{2}\Big(|0\rangle \big(|f(0)\rangle - |\bar{f}(0)\rangle \big) + |1\rangle \big(|f(1)\rangle - |\bar{f}(1)\rangle \big)\Big)$
4. This last expression can be factored depending on f:
   $U\frac{1}{2}\Big(|0\rangle (|0\rangle - |1\rangle) + |1\rangle (|0\rangle - |1\rangle)\Big) =$
   $\begin{cases} \frac{1}{2}\big(|0\rangle + |1\rangle \big)\big(|f(0)\rangle - |\bar{f}(0)\rangle \big) \text{ if } f(0) = f(1) \\ \frac{1}{2}\big(|0\rangle - |1\rangle \big)\big(|f(0)\rangle - |\bar{f}(0)\rangle \big) \text{ if } f(0) \neq f(1) \end{cases} = \begin{cases} |+\rangle |-\rangle \text{ if } f(0) = f(1) \\ |-\rangle |-\rangle \text{ if } f(0) \neq f(1) \end{cases}$

The trick where oracle's output on $|t\rangle$ affects phase of $|c\rangle$ is called phase kickback.

Demonstration of Deutsch-Jozsa for the $n = 1$ case

Output of circuit is $c = 0$ iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle \otimes |1\rangle = |0\rangle |1\rangle = |01\rangle$
2. After first set of Hadamards: $\frac{1}{2}\Big(|0\rangle (|0\rangle - |1\rangle) + |1\rangle (|0\rangle - |1\rangle) \Big)$
3. After applying oracle U:
$$U\frac{1}{2}\Big(|0\rangle (|0\rangle - |1\rangle) + |1\rangle (|0\rangle - |1\rangle) \Big) = \begin{cases} |+\rangle |-\rangle & \text{if } f(0) = f(1) \\ |-\rangle |-\rangle & \text{if } f(0) \neq f(1) \end{cases}$$
4. After applying second H on top qubit:
$$\begin{cases} H \otimes I\big(|+\rangle |-\rangle \big) = |0\rangle |-\rangle & \text{if } f(0) = f(1) \\ H \otimes I\big(|-\rangle |-\rangle \big) = |1\rangle |-\rangle & \text{if } f(0) \neq f(1) \end{cases}$$

Deutsch-Jozsa programs and systems

Algorithm

David Deutsch and Richard Jozsa. Rapid solution of problems by quantum computation. 1992.

Programs

Google Cirq programming example.

Implementation

- Mach-Zehnder interferometer implementation. `https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/SinglePhotonLab/SinglePhotonLab.html`
- Ion trap implementation. Gulde et al. Implementation of the Deutsch–Jozsa algorithm on an ion-trap quantum computer. Letters to Nature. 2003.

Mach-Zehnder interferometer implementation of Deutsch's algorithm

$$
|0\rangle \xrightarrow{H} |+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}
\begin{cases}
\xrightarrow{I} |+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} & \xrightarrow{H} |0\rangle \\
\xrightarrow{Z} |-\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} & \xrightarrow{H} |1\rangle \\
\xrightarrow{-Z} -|-\rangle = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} & \xrightarrow{H} -|1\rangle \\
\xrightarrow{-ZZ=-I} -|+\rangle = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} & \xrightarrow{H} -|0\rangle
\end{cases}
$$

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Deutsch-Jozsa algorithm: Deutsch's algorithm for the $n > 1$ case

The state after the first set of Hadamards

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle^{\otimes n} \otimes |1\rangle = |0 \ldots 0\rangle |1\rangle = |0 \ldots 01\rangle$
2. After first set of Hadamards: $|+\rangle^{\otimes n} \otimes |-\rangle = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} |c\rangle \otimes |-\rangle$

Deutsch's algorithm: Deutsch-Jozsa for the $n = 1$ case

The state after applying oracle U

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle^{\otimes n} \otimes |1\rangle = |0\ldots0\rangle\,|1\rangle = |0\ldots01\rangle$
2. After first set of Hadamards: $|+\rangle^{\otimes n} \otimes |-\rangle = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} |c\rangle \otimes |-\rangle$
3. After applying oracle U:

$$U\Big(|+\rangle^{\otimes n} \otimes |-\rangle\Big) = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} |c\rangle \otimes \left(\frac{|f(c)\rangle - |\bar{f}(c)\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

Lemma: the Hadamard transform

$H^{\otimes n}|c\rangle = \frac{1}{2^{n/2}} \sum_{m=0}^{2^n-1} (-1)^{c \cdot m} |m\rangle$

- $$\begin{aligned} &H^{\otimes n}|c\rangle \\ &= H|c_0\rangle \otimes H|c_1\rangle \otimes \ldots \otimes H|c_{n-1}\rangle \\ &= \frac{1}{\sqrt{2}}\Big(|0\rangle + (-1)^{c_0}|1\rangle\Big) \otimes \frac{1}{\sqrt{2}}\Big(|0\rangle + (-1)^{c_1}|1\rangle\Big) \otimes \ldots \otimes \frac{1}{\sqrt{2}}\Big(|0\rangle + (-1)^{c_{n-1}}|1\rangle\Big) \\ &= \frac{1}{2^{n/2}} \sum_{m=0}^{2^n-1} (-1)^{c_0 m_0 + c_1 m_1 + \ldots + c_{n-1} m_{n-1} \mod 2} |m\rangle \end{aligned}$$
- Try it out for $n = 1$: $H^{\otimes 1}|c\rangle = \frac{1}{2^{1/2}} \sum_{m=0}^{2^1-1} (-1)^{c \cdot m} |m\rangle =$
  $\frac{1}{\sqrt{2}}(-1)^0|0\rangle + \frac{1}{\sqrt{2}}(-1)^c|1\rangle = \begin{cases} \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = |+\rangle & \text{if } |c\rangle = |0\rangle \\ \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = |-\rangle & \text{if } |c\rangle = |1\rangle \end{cases}$

Deutsch-Jozsa algorithm: Deutsch's algorithm for the $n > 1$ case

The state after applying oracle U

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle^{\otimes n} \otimes |1\rangle = |0\ldots0\rangle\,|1\rangle = |0\ldots01\rangle$
2. After first set of Hadamards: $|+\rangle^{\otimes n} \otimes |-\rangle = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} |c\rangle \otimes |-\rangle$
3. After applying oracle U: $U\Big(|+\rangle^{\otimes n} \otimes |-\rangle\Big) = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$
4. After final set of Hadamards:

$$(H^{\otimes n} \otimes I)\left(\frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\right)$$

$$= \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} \left(\frac{1}{2^{n/2}} \sum_{m=0}^{2^n-1} (-1)^{c \cdot m} |m\rangle\right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{2^n} \sum_{c=0}^{2^n-1} \sum_{m=0}^{2^n-1} (-1)^{f(c) + c \cdot m} |m\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

Deutsch-Jozsa algorithm: Deutsch's algorithm for the $n > 1$ case

Output of circuit is 0 iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle^{\otimes n} \otimes |1\rangle = |0\dots0\rangle\,|1\rangle = |0\dots01\rangle$
2. After first set of Hadamards: $|+\rangle^{\otimes n} \otimes |-\rangle = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} |c\rangle \otimes |-\rangle$
3. After applying oracle U: $U\Big(|+\rangle^{\otimes n} \otimes |-\rangle\Big) = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$
4. After final set of Hadamards: $(H^{\otimes n} \otimes I)\left(\frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\right) =$
   $\frac{1}{2^n} \sum_{c=0}^{2^n-1} \sum_{m=0}^{2^n-1} (-1)^{f(c)+c\cdot m} |m\rangle \otimes \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$
5. Amplitude of upper register being $|m\rangle = |0\rangle$:

$$\frac{1}{2^n} \sum_{c=0}^{2^n-1} (-1)^{f(c)}$$

Deutsch-Jozsa algorithm: Deutsch's algorithm for the $n > 1$ case

Output of circuit is 0 iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle^{\otimes n} \otimes |1\rangle = |0\ldots 0\rangle\,|1\rangle = |0\ldots 01\rangle$
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3. After applying oracle U: $U\Big(|+\rangle^{\otimes n} \otimes |-\rangle\Big) = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$
4. After final set of Hadamards: $(H^{\otimes n} \otimes I)\left(\frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\right) =$
   $\frac{1}{2^n} \sum_{c=0}^{2^n-1} \sum_{m=0}^{2^n-1} (-1)^{f(c)+c\cdot m} |m\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$
5. Amplitude of upper register being $|m\rangle = |0\rangle$: $\frac{1}{2^n} \sum_{c=0}^{2^n-1} (-1)^{f(c)}$
6. Probability of measuring upper register to get $m = 0$:

$$\left| \frac{1}{2^n} \sum_{c=0}^{2^n-1} (-1)^{f(c)} \right|^2 = \begin{cases} \left|(-1)^{f(c)}\right|^2 = 1 & \text{if } f \text{ is constant} \\ 0 & \text{if } f \text{ is balanced} \end{cases}$$