Representing and Manipulating Information: Fixed point, floating point normalized and denormalized numbers

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Programming assignment 2

Programming assignment 2

1. Due Friday 2/23.
2. Graph algorithms and hash table.

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Representing negative and signed integers

Ways to represent negative numbers

1. Sign magnitude
2. 1s' complement
3. 2's complement

Representing negative and signed integers

2's complement

1) flip all bits
+1

2)

signed char	weight in decimal
00000001	1
00000010	2
00000100	4
00001000	8
00010000	16
00100000	32
01000000	64
10000000	-128

Table: Weight of each bit in a signed char type

- what is the most positive value you can represent? 127
- what is the most negative value you can represent? -128
- how to represent -1? 11111111
- how to represent -2? 11111110

Representing negative and signed integers

2's complement

signed char	weight in decimal
00000001	1
00000010	2
00000100	4
00001000	8
00010000	16
00100000	32
01000000	64
10000000	-128

Table: Weight of each bit in a signed char type

- MSB: 1 for negative
- To make a number negative: flip all bits and add 1.
- Addition in 2's complement is sound

Importance of paying attention to limits of encoding

15+15=30

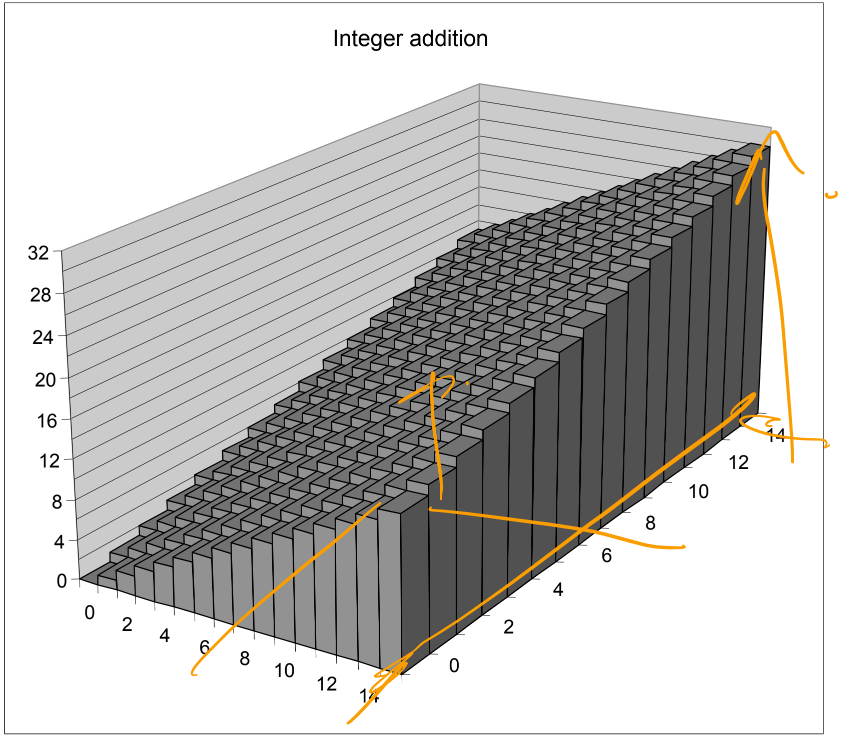


Figure: Image credit: CS:APP

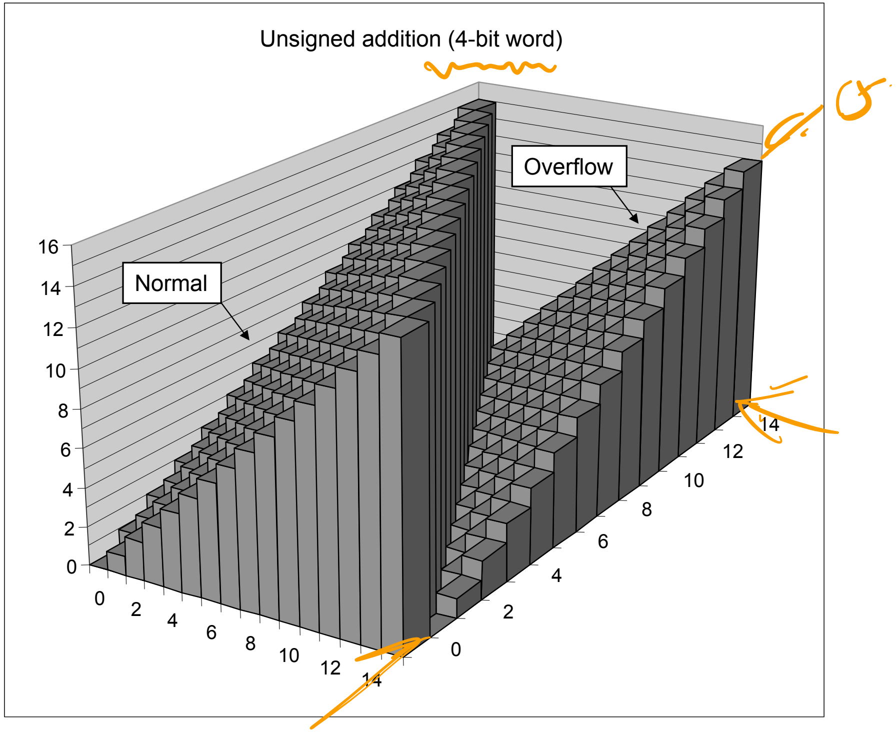
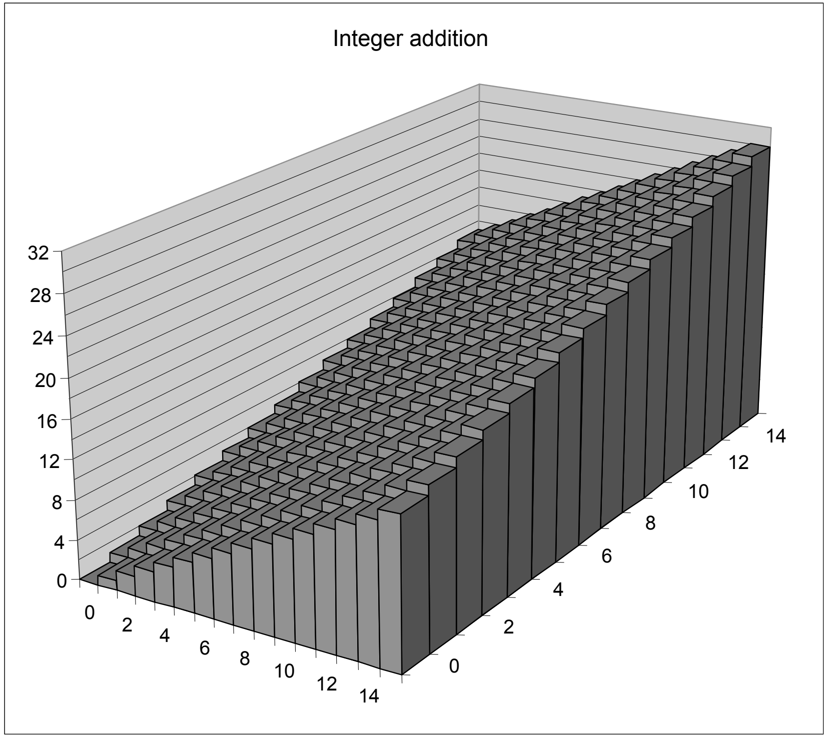


Figure: Image credit: CS:APP

Importance of paying attention to limits of encoding



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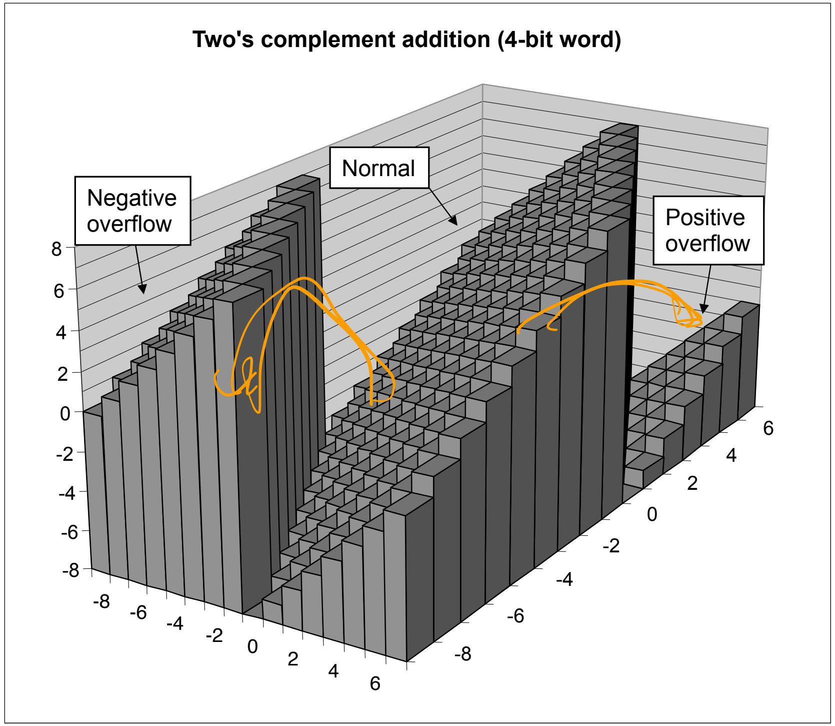

Figure: Image credit: CS:APP

https://www.theatlantic.com/technology/archive/2014/12/how-gangnam-style-broke-youtube/383389/

$$2^{32} = 2^{30+2} = 2^{30} \times 2^{2} = 2^{2} \times 2^{30}$$

$$= 4 \times 2^{(10+10+10)} = 4 \times \left(2^{10}\right)^{3}$$

$$\approx 4 \times (1000)^{3} = 4 \times 10^{9} = 4 \text{ Billion}$$

GB = gigabyte = 10^9 bytes

GiB = gigibyte = $(2^{10})^3$

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6.251

1s $\frac{1}{10}$ $\frac{1}{100}$ $\frac{1}{1000}$

06.251

6.2510000

2 ""

2.000000 ""

Unsigned fixed-point binary for fractions

fractional
decimal
scientific notation

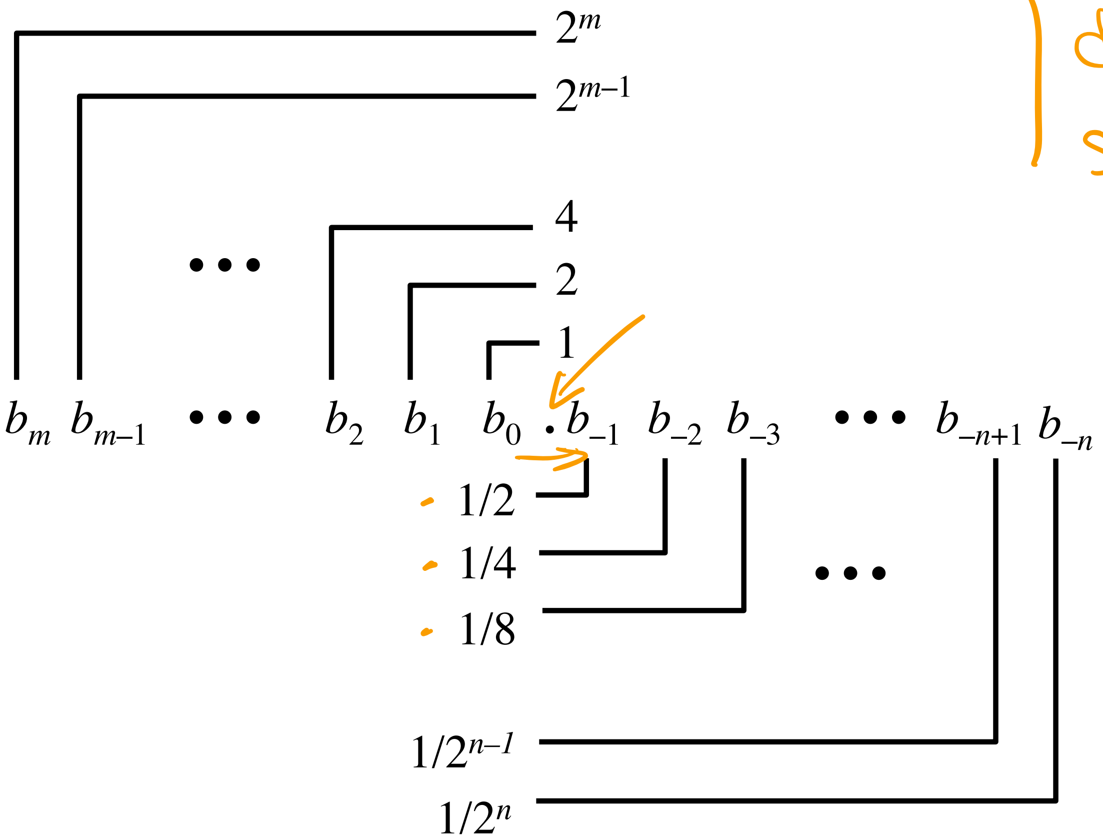

Figure: Fractional binary. Image credit CS:APP

Unsigned fixed-point binary for fractions

unsigned fixed-point char example	weight in decimal
1000.0000	8
0100.0000	4
0010.0000	2
0001.0000	1
0000.1000	0.5
0000.0100	0.25
0000.0010	0.125
0000.0001	0.0625

Table: Weight of each bit in an example fixed-point binary number

- $.625 = .5 + .125 = 0000.1010_2$
- $1001.1000_2 = 9 + .5 = 9.5$

$$3.14 \xrightarrow{\text{shift left by 1}} 31.4$$

$$31.4 = 3.14 \times 10^1$$

3.14_{10}

8-bit unsigned fixed point w/ binary point at 4 places from MSB

$$\begin{array}{cccccccccc} \_ & \_ & \_ & \_ & . & \_ & \_ & \_ & \_ \\ 8 & 4 & 2 & 1 & & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \end{array}$$

$$0011.0010 \longrightarrow 2+1+\frac{1}{8} = 3.125$$

$$\begin{array}{r} 0.14_{10} \\ -\ 0.125_{10} \\ \hline 0.015_{10} \\ 0.0625 \end{array}$$

$$\frac{1}{3} = 0.\overline{3}_{10}$$

$$0.\overline{1}_2 = 0.1\overline{1}_2 = 0.11\overline{1}_2$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$$

$$\rightarrow 1$$

Signed fixed-point binary for fractions

signed fixed-point char example	weight in decimal
1000.0000	-8
0100.0000	4
0010.0000	2
0001.0000	1
0000.1000	0.5
0000.0100	0.25
0000.0010	0.125
0000.0001	0.0625

Table: Weight of each bit in an example fixed-point binary number

- $-.625 = -8 + 4 + 2 + 1 + 0 + .25 + .125 = 1111.0110_2$
- $1001.1000_2 = -8 + 1 + .5 = -6.5$

Limitations of fixed-point

- Can only represent numbers of the form $x/2^k$
- Cannot represent numbers with very large magnitude (great range) or very small magnitude (great precision)

Bit shifting

$<< N$ Left shift by N bits

- multiplies by 2^N
- $2 << 3 = 0000\_0010_2 << 3 = 0001\_0000_2 = 16 = 2 * 2^3$
- $-2 << 3 = 1111\_1110_2 << 3 = 1111\_0000_2 = -16 = -2 * 2^3$

$>> N$ Right shift by N bits

- divides by 2^N
- $16 >> 3 = 0001\_0000_2 >> 3 = 0000\_0010_2 = 2 = 16/2^3$
- $-16 >> 3 = 1111\_0000_2 >> 3 = 1111\_1110_2 = -2 = -16/2^3$

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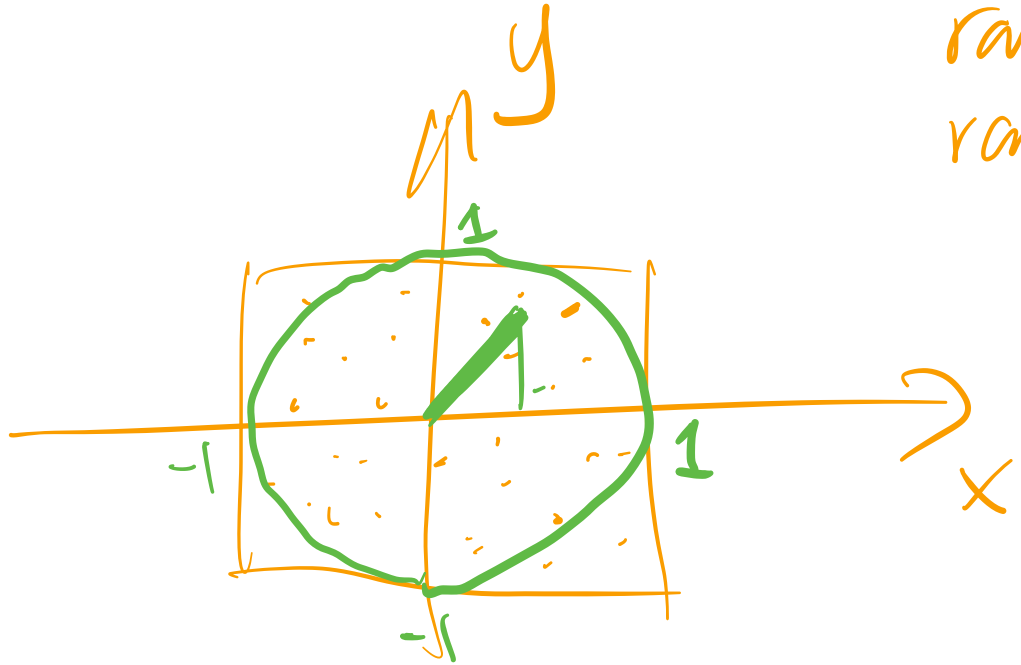
`monteCarloPi.c` Using floating point and random numbers to estimate PI

π



rand x
rand y

for (I to 100000) {

if ($\sqrt{randx^2 + randy^2} < 1$) {

add to tally inside

add tally

}

$\frac{\text{inside}}{\text{total}}$

$1 \cdot 1 \cdot \pi = \text{ratio} \cdot 1 \cdot 1 \cdot 4$

$\pi = 4 \cdot \text{ratio}.$

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Floating point numbers

Avogadro's number

$+6.02214 \times 10^{23} \, mol^{-1}$

Scientific notation

- sign
- mantissa or significand
- exponent

Floating point numbers

Before 1985

1. Many floating point systems.
2. Specialized machines such as Cray supercomputers.
3. Some machines with specialized floating point have had to be kept alive to support legacy software.

After 1985

1. IEEE Standard 754.
2. A floating point standard designed for good numerical properties.
3. Found in almost every computer today, except for tiniest microcontrollers.

Recent

1. Need for both lower precision and higher range floating point numbers.
2. Machine learning / neural networks. Low-precision tensor network processors.

Floats and doubles

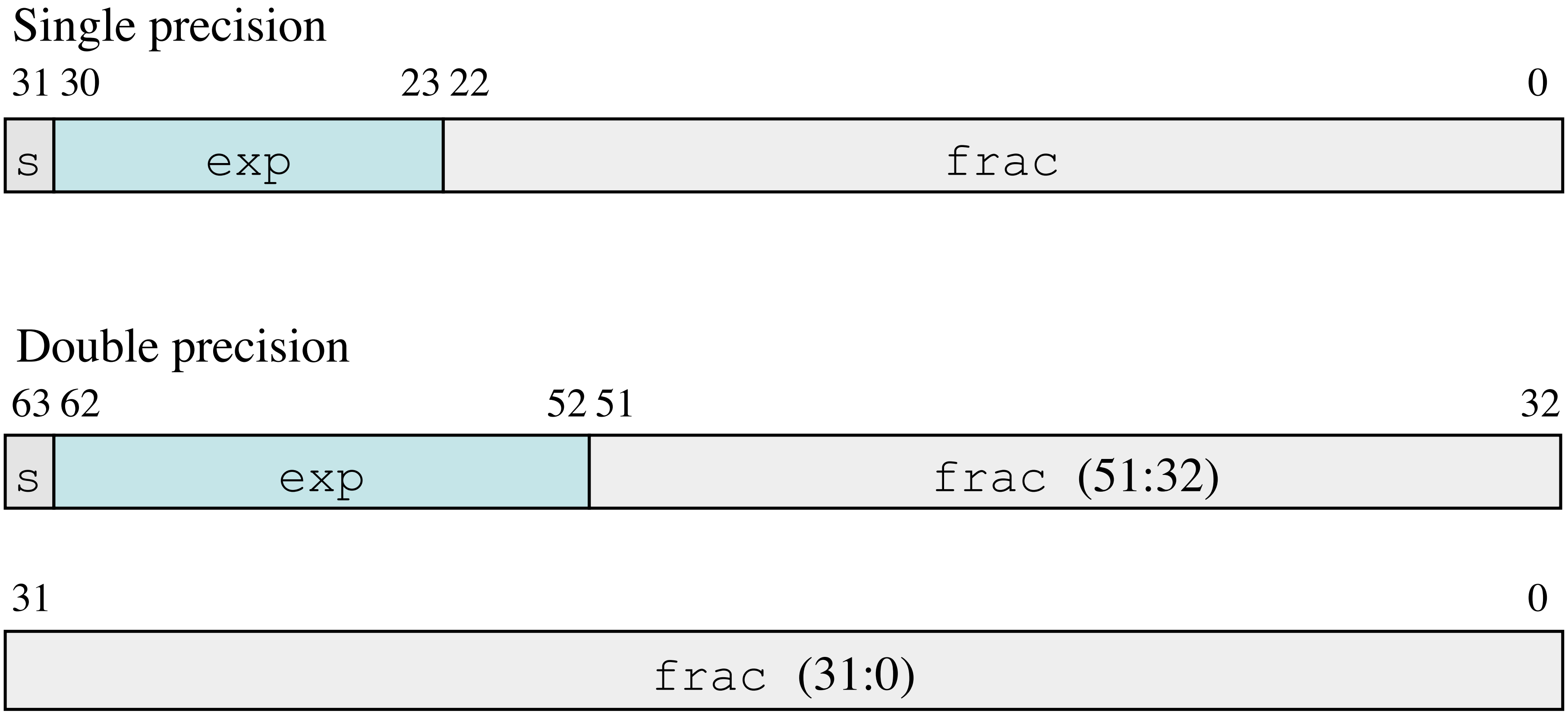

Figure: The two standard formats for floating point data types. Image credit CS:APP

Floats and doubles

property	half*	float	double
total bits	16	32	64
s bit	1	1	1
exp bits	5	8	11
frac bits	10	23	52
C printf() format specifier	None	"%f"	"%lf"

Table: Properties of floats and doubles

The IEEE 754 number line

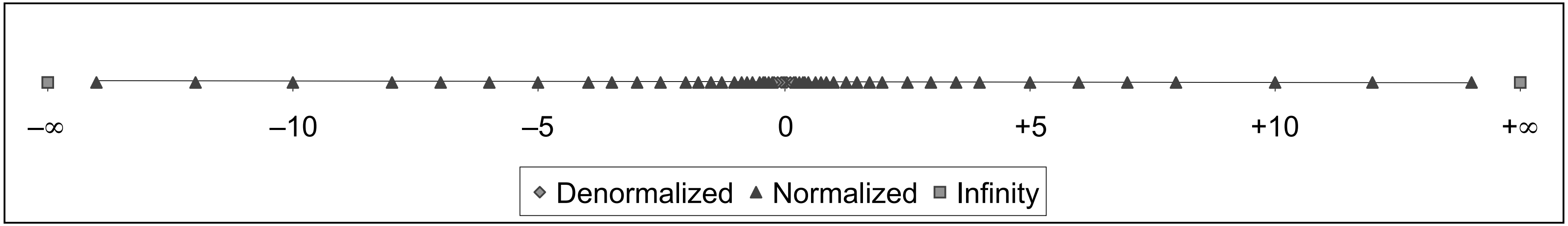

Figure: Full picture of number line for floating point values. Image credit CS:APP

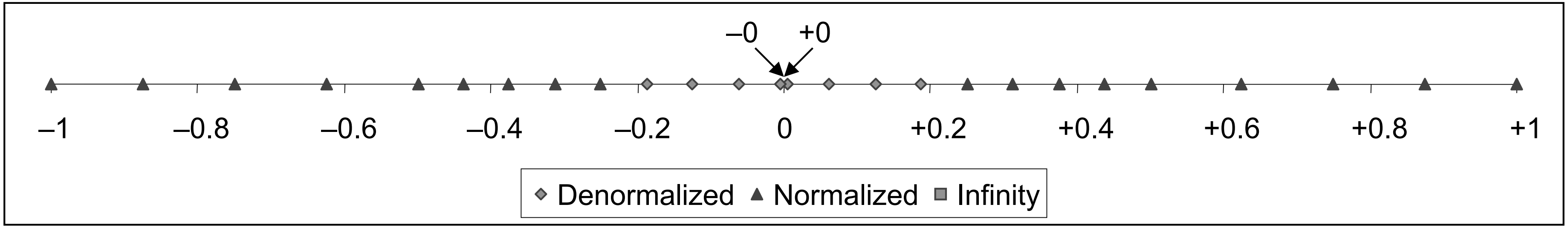

Figure: Zoomed in number line for floating point values. Image credit CS:APP

Different cases for floating point numbers

Value of the floating point number = $(-1)^s \times M \times 2^E$

- E is encoded the exp field
- M is encoded the frac field

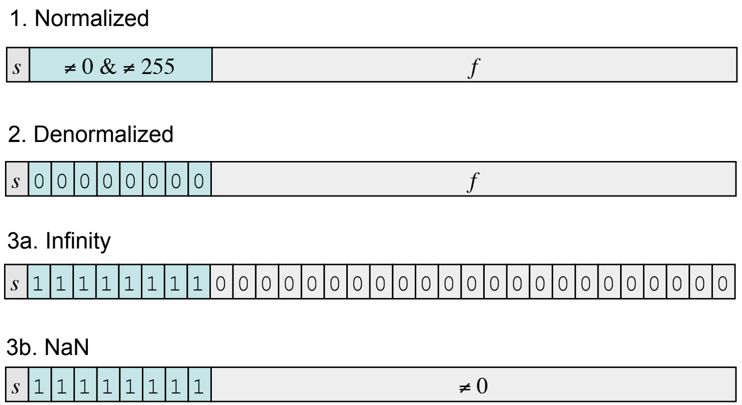

Figure: Different cases within a floating point format. Image credit CS:APP

Normalized and denormalized numbers

Two different cases we need to consider for the encoding of E, M

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Normalized: exp field

For normalized numbers,
$0 < \text{exp} < 2^k - 1$

- exp is a k-bit unsigned integer

Bias

- need a bias to represent negative exponents
- bias = $2^{k-1} - 1$
- bias is the k-bit unsigned integer: 011..111

For normalized numbers,
E = exp-bias

In other words, exp = E+bias

property	float	double
k	8	11
bias	127	1023
smallest E (greatest precision)	-126	-1022
largest E (greatest range)	127	1023

Table: Summary of normalized exp field

Normalized: frac field

M = 1.frac

Normalized: example

- 12.375 to single-precision floating point
- sign is positive so s=0
- binary is 1100.011_2
- in other words it is $1.100011_2 \times 2^3$
- $\text{exp} = E + \text{bias} = 3 + 127 = 130 = 1000\_0010_2$
- M = 1.100011_2 = 1.frac
- frac = 100011

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The IEEE 754 number line

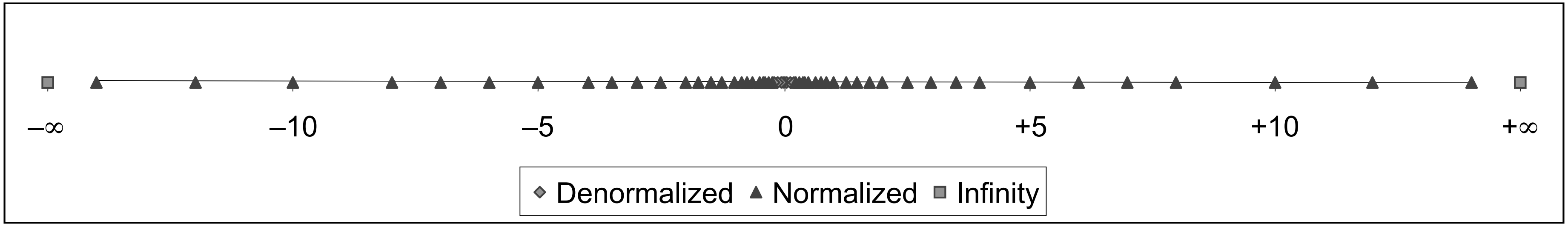

Figure: Full picture of number line for floating point values. Image credit CS:APP

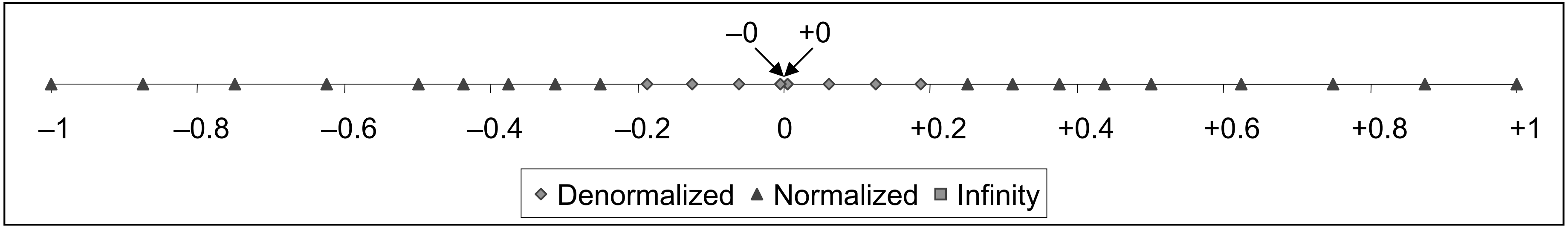

Figure: Zoomed in number line for floating point values. Image credit CS:APP

Denormalized: exp field

For denormalized numbers, $\exp = 0$

Bias

- need a bias to represent negative exponents
- bias = $2^{k-1} - 1$
- bias is the k-bit unsigned integer: 011..111

For denormalized numbers,
E = 1-bias

property	float	double
k	8	11
bias	127	1023
E	-126	-1022

Table: Summary of denormalized exp field

Denormalized: frac field

M = 0.frac

value represented leading with 0

Denormalized: examples

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Floats: Special cases

number class	when it arises	exp field	frac field
+0 / -0		0	0
+infinity / -infinity	overflow or division by 0	$2^k - 1$	0
NaN not-a-number	illegal ops. such as $\sqrt{-1}$, inf-inf, inf*0	$2^k - 1$	non-0

Table: Summary of special cases

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Floats: Summary

	normalized	denormalized
value of number	$(-1)^s \times M \times 2^E$	$(-1)^s \times M \times 2^E$
E	E = exp-bias	E = -bias + 1
bias	$2^{k-1} - 1$	$2^{k-1} - 1$
exp	$0 < exp < (2^k - 1)$	$exp = 0$
M	M = 1.frac	M = 0.frac
	M has implied leading 1	M has leading 0
	greater range	greater precision
	large magnitude numbers	small magnitude numbers
	denser near origin	evenly spaced

Table: Summary of normalized and denormalized numbers