Languages and representations for quantum computing: Stabilizer formalism

Yipeng Huang

Rutgers University

March 27, 2024
Quantum Circuit Simulation

What makes a circuit hard to simulate?

- # of qubits? (width)
- # of operators? (depth)

large superposition?

high fidelity?

large entanglement?
# non Cliffords
tree width
What is it that gives quantum computers an advantage compared to classical computing?

- Superposition?
- Entanglement?
- Both?
- Neither?
Importance of representations in quantum intuition, programming, and simulation

- Conventional quantum circuits and state vector view of QC conceals symmetries, hinders intuition.
- Classical simulation of quantum computing is actually tractable for *a certain subset* of quantum gates.
- Both the logical and native gatesets in a quantum architecture need to be *universal* for quantum advantage.
Several views/representations of quantum computing

- Programming has several views: functional programming, procedural programming.
- Physics has several views: Newtonian, Lagrangian, Hamiltonian

Different views reveal different symmetries, offer different intuition.
Several views/representations of quantum computing

- Schrödinger: state vectors and density matrices
- Heisenberg: stabilizer formalism
- Tensor-network
- Feynman: path sums

A survey of these representations of quantum computing is given in Chapter 9 of this recent book [Ding and Chong, 2020].
Several views/representations of quantum computing

- Schrödinger: state vectors and density matrices
- Heisenberg: stabilizer formalism
- Tensor-network
- Feynman: path sums
- Binary decision diagrams (new?)
- Logical satisfiability equations
Schrödinger view

- In Schrödinger quantum mechanics description, emphasis on how states evolve.

- \(CNOT_{0,1}(H_0 \otimes I_1) |00\rangle = CNOT_{0,1} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}

\[
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}
\]

- The Schrödinger view requires exponential storage: A quantum computer with \(N\) qubits can be in superposition of \(2^N\) basis states, requires \(2^N\) amplitudes to fully specify state.
Heisenberg view / stabilizer formalism

- In Heisenberg quantum mechanics description, emphasis on how operators evolve.
- If we limit operations to the Clifford gates (a subset of quantum gates), simulation tractable in polynomial time and space.
- Covers some quantum algorithms: quantum superdense coding, quantum teleportation, Deutsch-Jozsa, Bernstein-Vazirani, quantum error correction, most quantum error correction protocols.
- A model for probabilistic (but not quantum) computation.
Concrete example on Bell state circuit

\[ \text{CNOT}_{0,1}(H_0 \otimes I_1) |00\rangle \]

1. Start with \( N \) qubits with initial state \( |0\rangle^\otimes N \).
2. Represent the state as its group of stabilizers—\( |00\rangle : \{IZ, ZI\} \)
3. When simulating the quantum circuit, decompose the Clifford gates to stabilizer gates \( \{\text{CNOT}, H, P\} \).
4. Apply each of the stabilizer gates to the stabilizer representation.
   - Hadamard on first qubit—\( |+\rangle |0\rangle : \{IZ, XI\} \)
   - CNOT on both qubits—\( \frac{|00\rangle + |11\rangle}{\sqrt{2}} : \{ZZ, XX\} \)
Representing a state as its group of stabilizers

- A unitary operator $U$ stabilizes a pure state $|\psi\rangle$ if $U |\psi\rangle = |\psi\rangle$
Representing a state as its group of stabilizers

1. $I$ stabilizes everything.
2. $-I$ stabilizes nothing.
3. $X$ stabilizes $|+\rangle$: $X|+\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |+\rangle$

4. $-X$ stabilizes $|-\rangle$: $-X|-\rangle = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |-\rangle$

5. $Y$ stabilizes $|+i\rangle$: $Y|+i\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |+i\rangle$

6. $-Y$ stabilizes $|-i\rangle$: $-Y|-i\rangle = -\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |-i\rangle$

7. $Z$ stabilizes $|0\rangle$: $Z|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$

8. $-Z$ stabilizes $|1\rangle$: $-Z|1\rangle = -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$
Representing a state as its group of stabilizers

In other words,

1. $|0\rangle$ is stabilized by $\{I, Z\}$
2. $|1\rangle$ is stabilized by $\{I, -Z\}$
3. $|+\rangle$ is stabilized by $\{I, X\}$
4. $|-\rangle$ is stabilized by $\{I, -X\}$
5. $|+i\rangle$ is stabilized by $\{I, Y\}$
6. $|-i\rangle$ is stabilized by $\{I, -Y\}$
Special places on the Bloch sphere

\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \]
\[ = |\alpha| [\cos(\gamma) + i \cdot \sin(\gamma)] |0\rangle 
  + |\beta| [\cos(\gamma + \phi) + i \cdot \sin(\gamma + \phi)] |1\rangle \]
\[ = \cos\left(\frac{\theta}{2}\right)e^{i\gamma} |0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i(\gamma+\phi)} |1\rangle \]

Enforces \( |\alpha|^2 + |\beta|^2 = 1 \)

Figure: Bloch sphere showing pole states.
Source: Wikimedia.
Special places on the Bloch sphere

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

$$= |\alpha|[\cos(\gamma) + i \cdot \sin(\gamma)] |0\rangle$$

$$+ |\beta|[\cos(\gamma + \phi) + i \cdot \sin(\gamma + \phi)] |1\rangle$$

$$= \cos(\frac{\theta}{2})e^{i\gamma} |0\rangle + \sin(\frac{\theta}{2})e^{i(\gamma+\phi)} |1\rangle$$

Enforces $|\alpha|^2 + |\beta|^2 = 1$

Figure: Bloch sphere showing pole states.
Source: Wikimedia.
Representing a state as its group of stabilizers

For multi-qubit states, the group of stabilizers is the cartesian product of the single-qubit stabilizers

- $|00\rangle = |0\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, Z \otimes I, Z \otimes Z\}$
- $\frac{|00\rangle + |10\rangle}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle = |+\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, X \otimes I, X \otimes Z\}$
- $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ is stabilized by $\{I \otimes I, X \otimes X, -Y \otimes Y, Z \otimes Z\}$
Representing a state as its group of stabilizers

- Critical result from group theory: for any N-qubit stabilized state, only N elements needed to specify group—a result from abstract algebra group theory [Nielsen and Chuang, 2002, Appendix 2]
- So long as the quantum circuit consists only of Clifford gates, only N elements needed to specify whole quantum state.
- Contrast against $2^N$ amplitudes needed to specify a general N-qubit quantum state vector.
- For example a two-qubit states needs four amplitudes \( \{a_0, a_1, a_2, a_3\} \) to specify quantum state \( |\psi\rangle = a_0 |00\rangle + a_1 |01\rangle + a_2 |10\rangle + a_3 |11\rangle \).
Representing a state as its group of stabilizers

Critical result from group theory: for any N-qubit stabilized state, only N elements needed to specify group.

1. $|0\rangle$ is stabilized by $\{I, Z\}$, $Z$ is generator
2. $|1\rangle$ is stabilized by $\{I, -Z\}$, $-Z$ is generator
3. $|+\rangle$ is stabilized by $\{I, X\}$, $X$ is generator
4. $|-\rangle$ is stabilized by $\{I, -X\}$, $-X$ is generator
5. $|+i\rangle$ is stabilized by $\{I, Y\}$, $Y$ is generator
6. $|-i\rangle$ is stabilized by $\{I, -Y\}$, $-Y$ is generator
7. $|0\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, Z \otimes I, Z \otimes Z\}$, $\{I \otimes Z, Z \otimes I\}$ is generator
8. $|+\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, X \otimes I, X \otimes Z\}$, $\{I \otimes Z, X \otimes I\}$ is generator
9. $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ is stabilized by $\{I \otimes I, X \otimes X, -Y \otimes Y, Z \otimes Z\}$, $\{X \otimes X, Z \otimes Z\}$ is generator
Concrete example on Bell state circuit

\[ CNOT_{0,1}(H_0 \otimes I_1) |00\rangle \]

1. Start with \( N \) qubits with initial state \( |0\rangle^\otimes N \).
2. Represent the state as its group of stabilizers—\( |00\rangle : \{IZ, ZI\} \)
3. When simulating the quantum circuit, decompose the Clifford gates to stabilizer gates \( \{CNOT, H, P\} \).
4. Apply each of the stabilizer gates to the stabilizer representation.

- Hadamard on first qubit—\( |+\rangle |0\rangle : \{IZ, XI\} \)
- CNOT on both qubits—\( \frac{|00\rangle + |11\rangle}{\sqrt{2}} \): \( \{ZZ, XX\} \)
Stabilizer gates: \{CNOT, H, P\}

2. CNOT gate: induces entanglement.
3. Phase gate: induces complex phases. \( P = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \)

- Despite featuring superposition, entanglement, and complex amplitudes, is not universal for quantum computing.
- We shall see that the deeply symmetrical structure of these gates prevent access to full quantum Hilbert space.
Stabilizer gates are a generator for Pauli gates (i.e., Clifford gates decompose to stabilizer gates)

Pauli gates are rotations around respective axes by $\pi$.

- $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = PP$

- $X = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = HZH$

- $Y = iXZ$

- $X^2 = Y^2 = Z^2 = I$

- Symmetry is similar to quaternions.

- With Clifford gates consisting of $\{CNOT, H, P, I, X, Y, Z\}$, sufficient to build many quantum algorithms, including: quantum superdense coding, quantum teleportation, Deutsch-Jozsa, Bernstein-Vazirani, quantum error correction, most quantum error correction protocols.
Single qubit stabilizer gates bounce stabilizer states around an octahedron on the Bloch sphere

\[
P |0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle
\]

\[
P |1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = i |1\rangle
\]

\[
P |+\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{i} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = |+i\rangle
\]

\[
P |-\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{i} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = |-i\rangle
\]

\[
P |+i\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{i} \\ -\frac{1}{2} \end{bmatrix} = |\rangle
\]

\[
P |-i\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{i} \\ \frac{1}{2} \end{bmatrix} = |+\rangle
\]
Single qubit stabilizer gates bounce stabilizer states around an octahedron on the Bloch sphere

\[
H |0\rangle = \begin{bmatrix} 1 & 1 \\ \sqrt{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |+\rangle
\]

\[
H |1\rangle = \begin{bmatrix} 1 & 1 \\ \sqrt{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = |--\rangle
\]

\[
H |+\rangle = \begin{bmatrix} 1 & 1 \\ \sqrt{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle
\]

\[
H |--\rangle = \begin{bmatrix} 1 & 1 \\ \sqrt{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle
\]

\[
H |+i\rangle = \begin{bmatrix} 1 & 1 \\ \sqrt{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{2}i \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}}i \end{bmatrix} = |--\rangle
\]

\[
H |--i\rangle = \begin{bmatrix} 1 & 1 \\ \sqrt{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{2}i \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{\sqrt{2}}i \end{bmatrix} = |i\rangle
\]

\[
H |--i\rangle = \begin{bmatrix} 1 & 1 \\ \sqrt{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{2}i \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{\sqrt{2}}i \end{bmatrix} = |+i\rangle
\]
Apply each of the stabilizer gates to the stabilizer representation.

- **Phase:**
  1. $Z \rightarrow Z$
  2. $-Z \rightarrow -Z$
  3. $X \rightarrow Y$
  4. $-X \rightarrow -Y$
  5. $Y \rightarrow -X$
  6. $-Y \rightarrow X$

- **Hadamard:**
  1. $Z \rightarrow X$
  2. $-Z \rightarrow -X$
  3. $X \rightarrow Z$
  4. $-X \rightarrow -Z$
  5. $Y \rightarrow -Y$
  6. $-Y \rightarrow Y$
Single qubit stabilizer gates bounce stabilizer states around an octahedron on the Bloch sphere

**Figure:** Bloch sphere showing pole states. Source: Wikimedia.
Apply each of the stabilizer gates to the stabilizer representation.

» CNOT:

1. $X \otimes I \rightarrow X \otimes X$
2. $I \otimes X \rightarrow I \otimes X$
3. $Z \otimes I \rightarrow Z \otimes I$
4. $I \otimes Z \rightarrow Z \otimes Z$
Concrete example on Bell state circuit

\[ CNOT_{0,1}(H_0 \otimes I_1) |00\rangle \]

1. Start with N qubits with initial state \( |0\rangle^\otimes N \).
2. Represent the state as its group of stabilizers—\( |00\rangle : \{IZ, ZI\} \)
3. When simulating the quantum circuit, decompose the Clifford gates to stabilizer gates \( \{CNOT, H, P\} \).
4. Apply each of the stabilizer gates to the stabilizer representation.
   - Hadamard on first qubit—\( |+\rangle |0\rangle : \{IZ, XI\} \)
   - CNOT on both qubits—\( \frac{|00\rangle + |11\rangle}{\sqrt{2}} \): \( \{ZZ, XX\} \)
Gottesman-Knill theorem and its implications

- The Gottesman-Knill theorem states that there exists a classical algorithm that simulates any stabilizer circuit in polynomial time.
- Any quantum state created by a Clifford circuit, even if it has lots of superpositions and entanglement, is easy to classically simulate.
- Quantum computers need at least one non-Clifford gate to achieve universal quantum computation.
- The T gate, where $TT = P$, $PP = Z$ is one common choice.
- There are results showing that a quantum circuit is only exponentially hard to simulate w.r.t. the number of T-gates.
Main sources: [Gottesman, 1998] [Aaronson, ]

Further reference on separation of probabilistic and quantum computing: [Van Den Nes, 2010]

Further reference on applications in classical simulation of Clifford quantum circuits: [Aaronson and Gottesman, 2004]

Further reference on applications in classical simulation of general quantum circuits: [Bravyi and Gosset, 2016]
References

Aaronson, S.
Lecture 28, tues may 2: Stabilizer formalism.

Improved simulation of stabilizer circuits.

Improved classical simulation of quantum circuits dominated by clifford gates.

Quantum computer systems: Research for noisy intermediate-scale quantum computers.

The heisenberg representation of quantum computers.

Quantum computation and quantum information.

Classical simulation of quantum computation, the gottesman-knill theorem, and slightly beyond.