Simulating Quantum Computing with Stabilizers Beyond Qubits

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Abstract

Quantum computers have novel applications in cryptography and simulation of materials but currently suffer from considerable noise.

Our research studies the error-correcting capabilities of **qudits**, the generalization of a qubit, through **stabilizer codes.** Such codes are powerful yet efficiently simulable via **stabilizer circuit simulators** on classical hardware.

Until now, no qudit simulator exists despite the ubiquity of stabilizer codes and the need to benchmark different constructions under various system-scale loads.

We introduce **the first qudit stabilizer circuit simulator** capable of simulating qudits of prime dimension.

The difficulty of qudit simulation

- Qubits have 2 complex values while qudits have $\boldsymbol{d}$ complex values. These vectors represent the **state** $|\psi\rangle$ of the system.

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{n=1} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_{n=1} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_d \end{bmatrix}_{n=1} \rightarrow 2^n, 3^n, d^n \text{ complex numbers}$$

- Representing $\boldsymbol{n}$ qudits leads to an exponential representation
- States $|\psi\rangle$ can be changed by multiplying the state with a $d^n \times d^n$ matrix $\hat{O}$ called an **operator** or **quantum gate.**

Why is simulating quantum computing hard?

d \ n	2	4	8	16
2	128 B	2 kB	512 kB	32 GB
3	648 B	51.26 kB	328.42 MB	13.17 PB
5	4.88 kB	2.98 MB	1.11 TB	157.77 ZB
7	51.26 kB	328.42 MB	241.8 TB	7.14 RB

Figure 1: RAM for single precision $d^n \times d^n$ matrix ($2 \cdot 4d^{2n}$ bytes)

- These are huge matrices. But it turns out, we can efficiently simulate a subset of states, stabilizer states.

Background on stabilizer states

- **Stabilizer states:** the set of states one can reach by applying *any sequence of quantum gates* from $\{H, P, CNOT\}$ to state $|0\rangle$. The gates are called **Clifford gates.**
- Every $\boldsymbol{n}$-qubit state can be represented by a binary table called a **tableau**. The celebrated **Gottesman-Knill theorem** demonstrates how tracking the quantum state with a tableau is $O(n^2)$ when using only Clifford gates.
- Our work extends the binary tableau to an integer tableau for $\boldsymbol{n}$-qudit stabilizer states.

Visualizing stabilizer states for a single qubit

Wait what's the Bloch sphere?

- Visualizes transformation of complex values to rotations on a sphere.
- The closer you are to $|0\rangle$ or $|1\rangle$, the more likely you are to measure them.
- Stabilizer codes essentially "round" the state $|\psi\rangle$ to a stabilizer state, the red dots on the sphere
- Clifford gates can only move you between the red dots

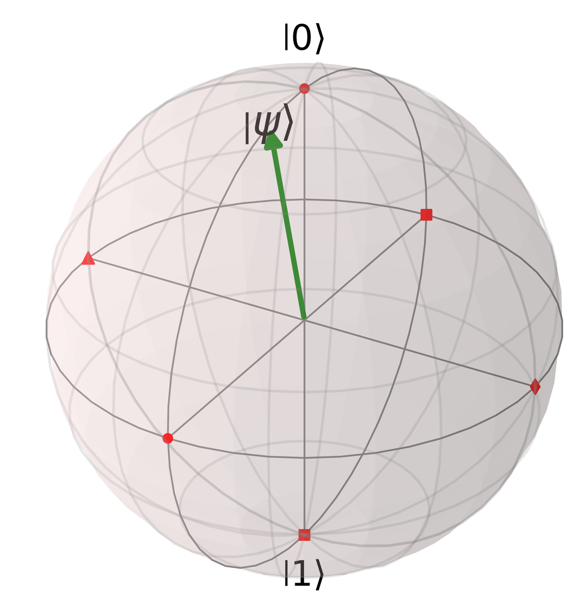
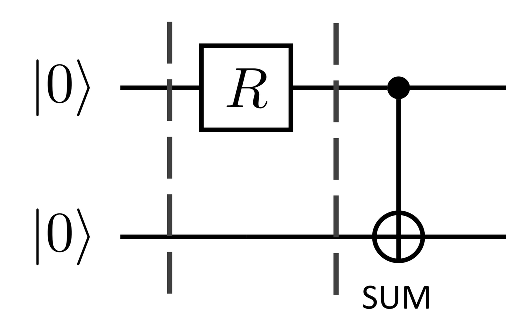


Figure 2: Bloch Sphere with random state $|\psi\rangle$ and Pauli stabilizers states plotted as red points

Example simulation of the qutrit Bell's state

$$|00\rangle_{d=3} \rightarrow \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$$



Clifford update rules

R	P
X → Z	$X \rightarrow XZ$
$Z \rightarrow X^{-1}$	$Z \rightarrow Z$

SUM
X ⊗ I → X ⊗ X
$I \otimes X \rightarrow I \otimes X$
$Z \otimes I \rightarrow Z \otimes I$
I ⊗ Z → Z^{-1} ⊗ Z

Figure 3: Conjugation table for R, P, SUM whose combinations produce every Clifford gate

Applying the Quantum Circuit

- The stabilizers are rows (3) and (4), but we also track "destabilizers" (1) and (2) to speed up the program, shown in the first column.
- Stabilizers keep $|\psi\rangle$ on the same point
- Destabilizers move $|\psi\rangle$ to the opposite point (ex. north → south pole).
- In the Tableau, we update the destabilizers and stabilizers according to the Clifford update rules.

Tableau

	Initial $|00\rangle$	Apply R	Apply SUM
1	$X \otimes I$	$Z \otimes I$	$Z \otimes I$
2	$I \otimes X$	$I \otimes X$	$I \otimes X$
3	$Z \otimes I$	$X^2 \otimes I$	$X^2 \otimes X^2$
4	$I \otimes Z$	$I \otimes Z$	$Z^{2} \otimes Z$

Figure 4: Tracking Tableau for Qutrit Bell State Circuit

Measuring the qudits in computational basis $(|0\rangle, |1\rangle, |2\rangle)$

- Tableau already has a single stabilizer (3) which doesn't commute with the measurement $Z \otimes$ I so we can directly measure qudit 1.
- We obtain a random eigenvalue $\{1, \omega, \omega^2\}$ which corresponds to $\{|0\rangle, |1\rangle, |2\rangle\}$ respectively. We then replace XX with $Z \otimes$ I multiplied by the negative of the measurement outcome.

	Measured $+1 \rightarrow |0\rangle$	Measured $\omega \rightarrow |1\rangle$	Measured $\omega^2 \rightarrow |2\rangle$
1	X^2X^2	X^2X^2	X^2X^2
2	IX	IX	IX
3	ZI	$\omega^{-1}ZI$	$\omega^{-2}ZI$
4	Z^2Z	Z^2Z	Z^2

Figure 5: Tableau after measuring first qudit

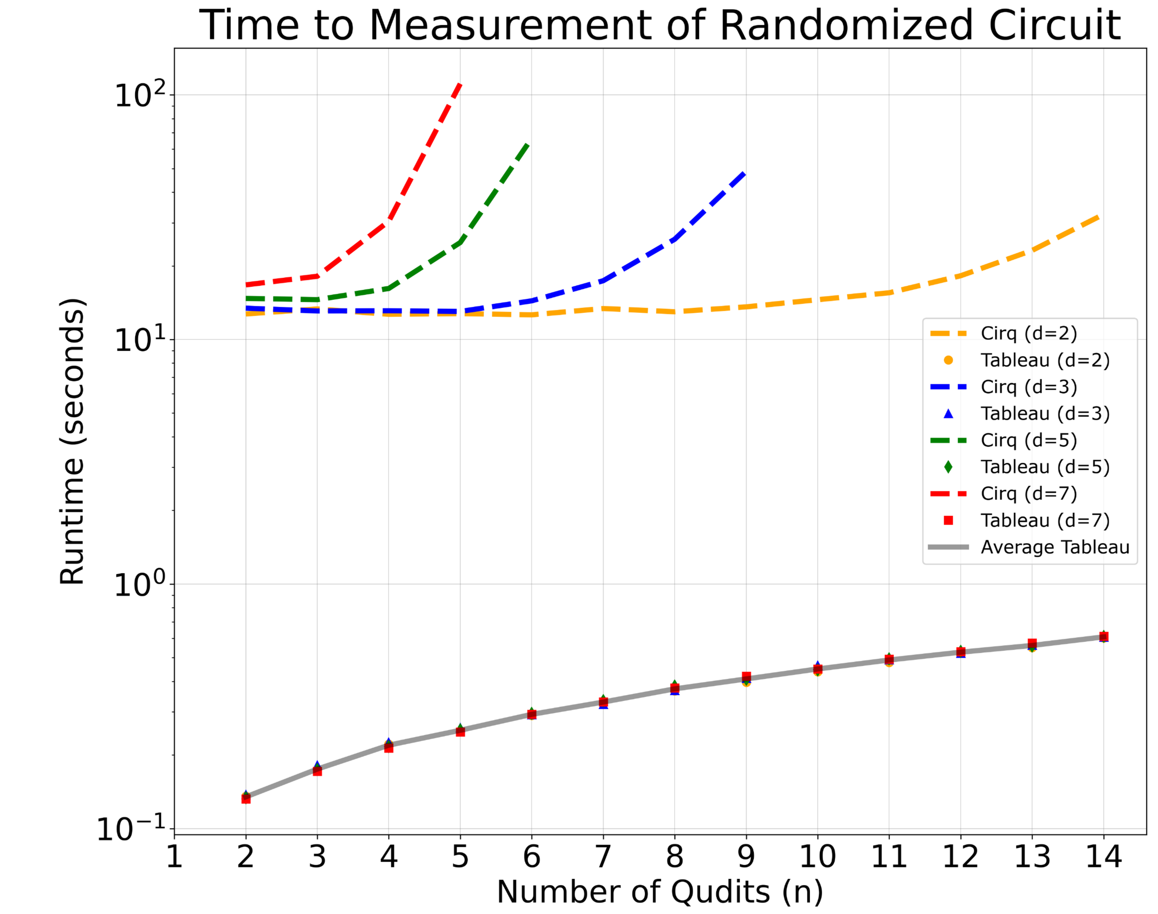
- Final step is measuring I ⊗ Z, the second qudit, which requires multiplying (3) and (4) together since destabilizers (1) and (2) contain X on qudit 2. This gives us the correlated value we expect as the second qubit is always equal to the first.

The key trick

- By tracking additional operators alongside stabilizers, we avoid matrix inversion during measurement
- We also showed that tableau operations for the generalized Pauli stabilizers become integer arithmetic modulo d for prime $d > 2$, instead of bitwise operations used in $d = 2$.

Scalability of our simulator

- Tested on circuits of 100,000 random gates consisting of 40% SUM, 30% R, 30% P gates
- Performance is polynomial in qudit count compared to statevector simulation's exponential performance
- Note that the tableau performance is independent of the qudit dimension



Future Direction

- Move from a standalone Python implementation to integrating with an existing stabilizer simulator such as Stim, QuantumClifford.jl, or even a more general quantum computing simulation platform like Qiskit and Cirq.
- Leverage the above platforms' mature and extensive noise profiling tools for thorough error-correction benchmarking and empirical study.
- Publicize the tool as a contribution toward the growing landscape of practical qudit research from theoretical study to concrete applications.

References

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