The Approximation of Density Matrices

Zirui Li, Yipeng Huang
Rutgers University
Circuit Cutting

Circuit Running:
Deploy the subcircuit on different devices.

Quantum Processor
Quantum Processor
Quantum Processor
Classical Processor
Classical Processor

Circuit Cutting

Subcircuit 1
Subcircuit 2

|0⟩
|0⟩
|0⟩
|0⟩

Measure
Measure

Initialize
Initialize

Initialize

Analysis on Density Matrices:
1) Noise-free density matrices:
   · Tensors are sparse!
2) Noisy density matrices:
   · Approximate by truncating the small values. Tensors are still sparse.

Tensor 1

Tensor 2

Postprocessing: contract tensor1 and tensor2

<table>
<thead>
<tr>
<th>q0</th>
<th>m1</th>
<th>m2</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>I</td>
<td>0.125</td>
</tr>
<tr>
<td>I</td>
<td>I</td>
<td>I</td>
<td></td>
</tr>
<tr>
<td>Z</td>
<td>I</td>
<td>X</td>
<td>0.2</td>
</tr>
<tr>
<td>I</td>
<td>I</td>
<td>I</td>
<td></td>
</tr>
<tr>
<td>Z</td>
<td>I</td>
<td>Y</td>
<td>0.12</td>
</tr>
</tbody>
</table>

ASPLOS’ 21 CutQC  Wei Tang et al.
Example: Cut the Bell State Circuit

The density matrix for subcircuit 1 is:

\[
\frac{1}{2}\left(|0\rangle + |1\rangle\right) \frac{1}{2}
\]

\[
= \frac{1}{2} I + \frac{1}{2} X
\]

denote the cutting point as m

The tensor for subcircuit 1 is:

\[
\begin{array}{cccc}
m & q_0 & q_1 & \text{value} \\
I & I & I & 0.5 \\
I & Z & Z & 0.5 \\
X & X & X & 0.5 \\
X & Y & Y & 0.5 \\
Y & Y & X & 0.5 \\
Y & X & Y & 0.5 \\
Z & I & Z & 0.5 \\
Z & Z & I & 0.5 \\
\end{array}
\]

The tensor for subcircuit 2 is (see top right Tensor 2).

Contract these 2 tensors you will get the same tensor from the Bell state density matrix.
Example: Bell State Density Matrix

• Density matrix of the Bell state \( \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \frac{1}{\sqrt{2}} (\langle00| + \langle11|) = \)

\[
\begin{bmatrix}
0.5 & 0 & 0 & 0.5 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0.5 & 0 & 0 & 0.5
\end{bmatrix}
= \frac{1}{4} II + \frac{1}{4} XX - \frac{1}{4} YY + \frac{1}{4} ZZ
\]

• Tensor contraction example:

eliminate variable \( j \)

\( C_{ik} = \sum_j A_{ij} B_{jk} \)

Contract Tensor 1 and Tensor 2: eliminate variable \( m \)

Tensor 1

<table>
<thead>
<tr>
<th>m</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0.5</td>
</tr>
<tr>
<td>X</td>
<td>0.5</td>
</tr>
<tr>
<td>Y</td>
<td>0</td>
</tr>
<tr>
<td>Z</td>
<td>0</td>
</tr>
</tbody>
</table>

Tensor 2

<table>
<thead>
<tr>
<th>m</th>
<th>q0</th>
<th>q1</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>I</td>
<td>0.5</td>
</tr>
<tr>
<td>X</td>
<td>Z</td>
<td>Z</td>
<td>0.5</td>
</tr>
<tr>
<td>Y</td>
<td>X</td>
<td>X</td>
<td>0.5</td>
</tr>
<tr>
<td>X</td>
<td>Y</td>
<td>Y</td>
<td>-0.5</td>
</tr>
<tr>
<td>Y</td>
<td>X</td>
<td>Y</td>
<td>0.5</td>
</tr>
<tr>
<td>Z</td>
<td>I</td>
<td>Z</td>
<td>0.5</td>
</tr>
<tr>
<td>Z</td>
<td>Z</td>
<td>I</td>
<td>0.5</td>
</tr>
<tr>
<td>otherwise</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Tensor Sparsity Analysis

VQE

QFT

QAOA

Noise-free

Noisy
Sparse Tensor Performance Analysis

We found that for tensor of size $N$, the sparsity is usually $\frac{\sqrt{N}}{N}$.

The number of multiplications for different operations can improve a lot.

<table>
<thead>
<tr>
<th></th>
<th>unitary gate</th>
<th>measure on an observable</th>
<th>tensor product</th>
<th>partial trace</th>
</tr>
</thead>
<tbody>
<tr>
<td>normal tensor</td>
<td>$2^{2N}$</td>
<td>$2^{2N}$</td>
<td>$2^{2N+2M}$</td>
<td>$2^{2N+M}$</td>
</tr>
<tr>
<td>sparse tensor</td>
<td>$2^N$</td>
<td>$2^N N$</td>
<td>$2^{N+M}$</td>
<td>$2^{N+M}$</td>
</tr>
</tbody>
</table>
Thank you!