The Approximation of Density Matrices

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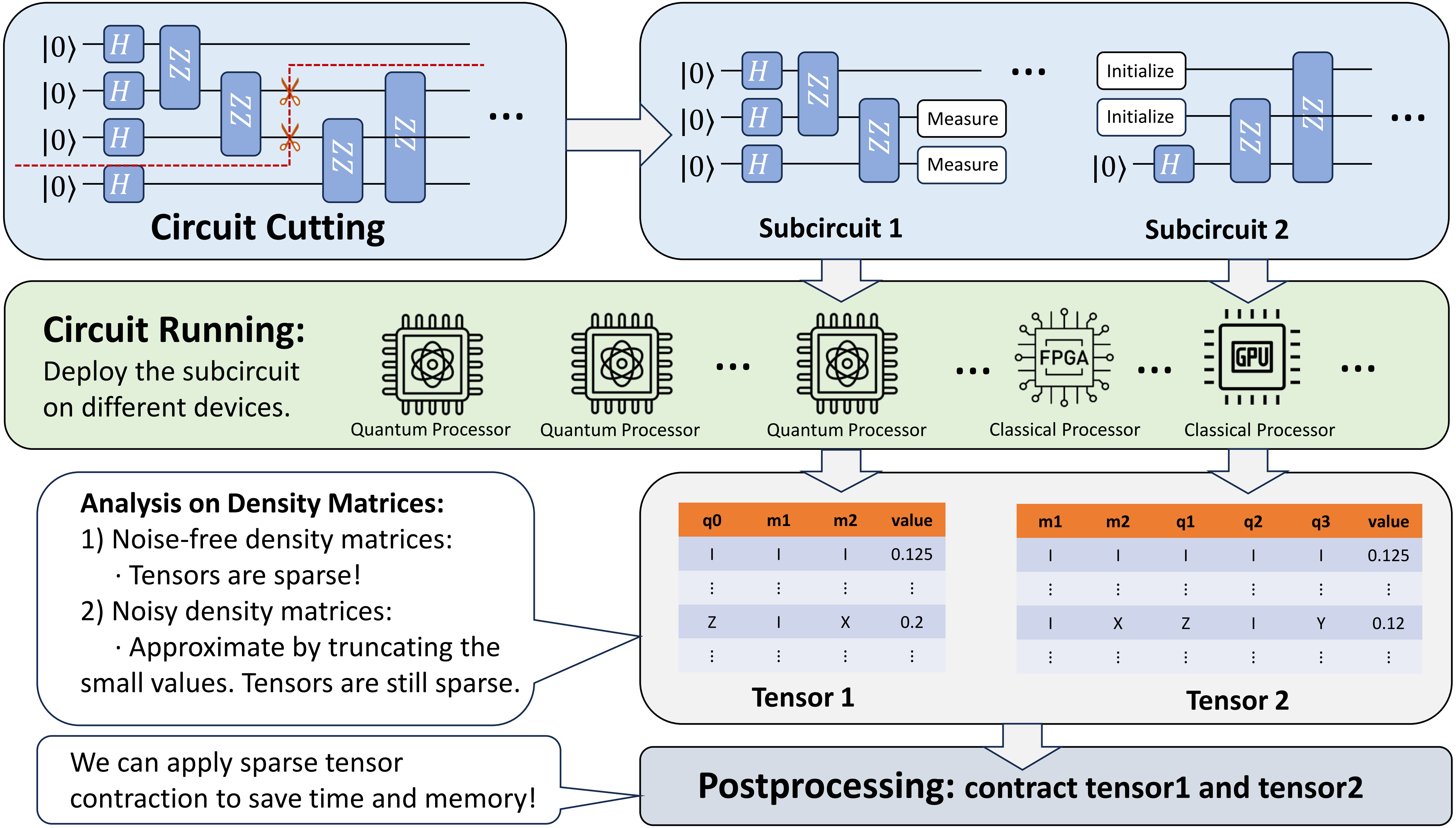
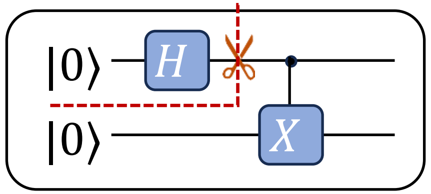
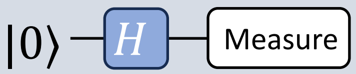
[ASPLOS' 21] CutQC Wei Tang et al.

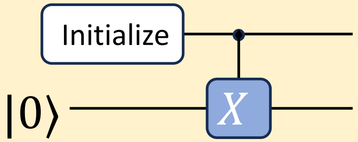


Example: Cut the Bell State Circuit





Subcircuit 1,



Subcircuit 2

m	q0	q1	value
I	I	I	0.5
I	Z	Z	0.5
X	X	X	0.5
X	Y	Y	-0.5
Y	Y	X	0.5
Y	X	Y	0.5
Z	I	Z	0.5
Z	Z	I	0.5
	otherwise		0

Tensor 2

The density matrix for subcircuit 1 is:

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \frac{1}{2}I + \frac{1}{2}X$$

denote the cutting point as m

The tensor for subcircuit1 is:

m	value
I	0.5
X	0.5
Y	0
Z	0

Tensor 1

After cut, we need to initialize 4 different state for m: $|0\rangle, |1\rangle, |+\rangle, |i\rangle$.

① When initial state is $|0\rangle$, $\mathrm{m} = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}I + \frac{1}{2}Z$. The output density is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{1}{4}II + \frac{1}{4}IZ + \frac{1}{4}ZI + \frac{1}{4}ZZ$

② When initial state is $|1\rangle$, $\mathrm{m} = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}I - \frac{1}{2}Z$. The output density is $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{4}II - \frac{1}{4}IZ - \frac{1}{4}ZI + \frac{1}{4}ZZ$

③ When initial state is $|+\rangle$, $\mathrm{m} = |+\rangle\langle +| = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \frac{1}{2}I + \frac{1}{2}X$. The output density is $\begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix} = \frac{1}{4}II + \frac{1}{4}XX - \frac{1}{4}YY + \frac{1}{4}ZZ$

④ When initial state is $|i\rangle$, $\mathrm{m} = |i\rangle\langle i| = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} = \frac{1}{2}I + \frac{1}{2}Y$. The output density is $\begin{bmatrix} 1/2 & 0 & 0 & -1/2\, i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2\, i & 0 & 0 & 1/2 \end{bmatrix} = \frac{1}{4}II + \frac{1}{4}YX + \frac{1}{4}XY + \frac{1}{4}ZZ$

Add ① and ②, we get ⑤ when $\mathrm{m} = I$, the output is $\frac{1}{2}II + \frac{1}{2}ZZ$; ③*2-⑤, we get ⑥ when $\mathrm{m} = X$, the output is $\frac{1}{2}XX - \frac{1}{2}YY$; ④*2-⑤, we get ⑦ when $\mathrm{m} = Y$, the output is $\frac{1}{2}YX + \frac{1}{2}XY$; ①-②, we get ⑧ when $\mathrm{m} = Z$, the output is $\frac{1}{2}IZ + \frac{1}{2}ZI$;

The tensor for subcircuit2 is (see top right Tensor 2).

Contract these 2 tensors you will get the same tensor from the Bell state density matrix.

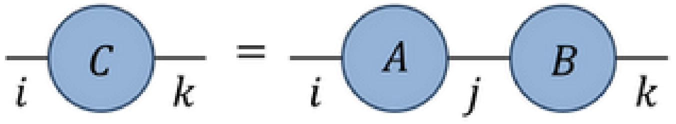
Example: Bell State Density Matrix

- Density matrix of the Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)$=

$$\begin{bmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{bmatrix} = \frac{1}{4}II + \frac{1}{4}XX - \frac{1}{4}YY + \frac{1}{4}ZZ$$

- Tensor contraction example:

  eliminate variable j



$$\Updownarrow$$

$$C_{ik} = \sum_j A_{ij}B_{jk}$$

q0	q1	value
I	I	0.25
X	X	0.25
Y	Y	-0.25
Z	Z	0.25
Otherwise		0

Tensor of the Bell State

contraction

Contract Tensor 1 and Tensor 2: eliminate variable m

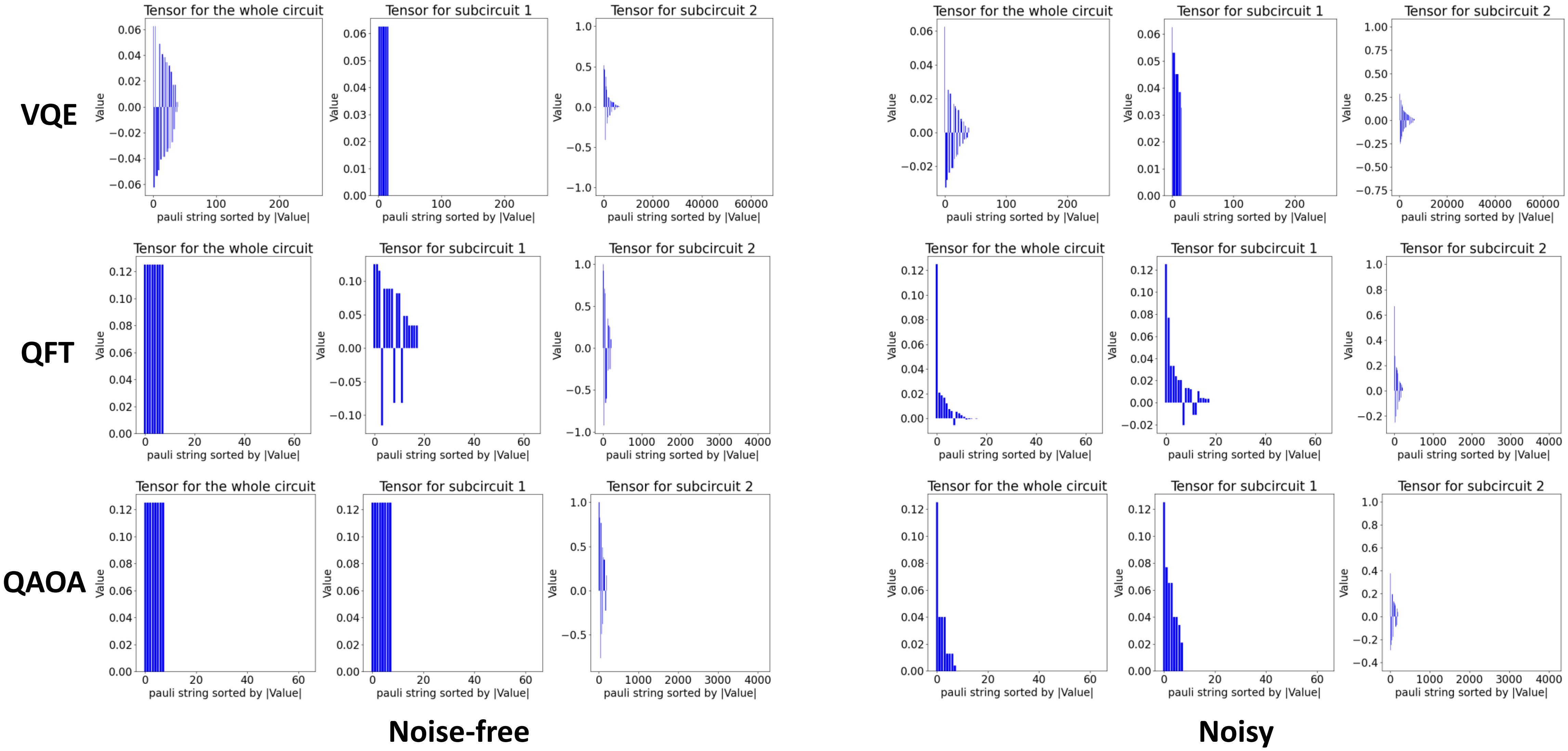
m	value
I	0.5
X	0.5
Y	0
Z	0

Tensor 1

m	q0	q1	value
I	I	I	0.5
I	Z	Z	0.5
X	X	X	0.5
X	Y	Y	-0.5
Y	Y	X	0.5
Y	X	Y	0.5
Z	I	Z	0.5
Z	Z	I	0.5
otherwise			0

Tensor 2

Tensor Sparsity Analysis



Sparse Tensor Performance Analysis

We found that for tensor of size N, the sparsity is usually $\frac{\sqrt{N}}{N}$.

The number of multiplications for different operations can improve a lot.

	unitary gate	**measure on an observable**	**tensor product**	**partial trace**
normal tensor	2^{2N}	2^{2N}	2^{2N+2M}	2^{2N+M}
sparse tensor	2^{N}	$2^{N}N$	2^{N+M}	2^{N+M}

Thank you!