Quantum computing fundamentals: States, Noisy States, Composition, Dynamics

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Postulates of quantum mechanics

1. State space
2. Composite systems
3. Evolution
4. Quantum measurement

1, 2, and 3 are linear and describe closed quantum systems. 4 is nonlinear and describes open quantum systems.

One Qubit.

→ Pure States. Bloch Sphere.

→ Noisy States.

→ Stabilizer States

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Exercise: The Qubit's Gambit

- What is the qubit state represented by the geographical location of Old Queens?

Quantum postulate 1: State space

The position or momentum of a physical system is described as a wavefunction

- Assuming continuous state space:

$$|\psi\rangle = \int_{-\infty}^{\infty} \psi(x) \, |x\rangle \, dx$$

$$|x\rangle$$

are orthonormal

$$\psi(x) \in \mathbb{C}$$

- Assuming discrete state space:

$$|\psi\rangle = \sum_{i=0}^{\infty} \psi(x_i) \, |x_i\rangle$$

$$\psi(x) \in \mathbb{C}$$

Quantum postulate 1: State space

The position or momentum of a physical system is described as a wavefunction

- Assuming discrete state space:

$$|\psi\rangle = \sum_{i=0}^{\infty} \psi(x_i) |x_i\rangle$$

$$\psi(x) \in \mathbb{C}$$

- Assuming discrete binarized state space:

$$|\psi\rangle = \sum_{i=0}^{1} \psi(x_i) |x_i\rangle$$

$$\psi(x) \in \mathbb{C}$$

The binary abstraction

High, low voltage

Adds resilience against noise.

Representation as a state vector

- $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$ We pronounce this "ket" 0
- $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$ We pronounce this "ket" 1

The NOT gate

Matrix representation of NOT operator: $X = \sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

- $X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$
- $X|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$

Circuit diagram representation:

The Hadamard gate

Matrix representation of Hadamard operator: $H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

- $H\left|0\right\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle$
- $H\left|1\right\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{1}{\sqrt{2}}\left|1\right\rangle$

Circuit diagram representation:

Interference

Amplitudes can positively and negatively interfere

- $HH\left|0\right\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left|0\right\rangle$
- $HH\left|1\right\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2} \\ \frac{1}{2} + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left|1\right\rangle$

Circuit diagram representation:

Superposition

Single qubit state

- $\alpha\,|0\rangle + \beta\,|1\rangle = \begin{bmatrix}\alpha \\ \beta\end{bmatrix}$
- Amplitudes $\alpha, \beta \in \mathbb{C}$
- $|\alpha|^2 + |\beta|^2 = 1$
- The above constraints require that qubit operators are unitary matrices.

Many physical phenomena can be in superposition and encode qubits

- Polarization of light in different directions
- Electron spins (Intel solid state qubits)
- Atom energy states (UMD, IonQ ion trap qubits)
- Quantized voltage and current (IBM, Google superconducting qubits)

If multiple discrete values are possible (e.g., atom energy states, voltage and current), we pick (bottom) two for the binary abstraction.

The phase shift gate

Matrix representation of phase shift operator: $Z = \sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

- $Z|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$
- $Z|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -|1\rangle$

Circuit diagram representation:

A simple physics experiment that classical computing cannot replicate

Algorithm

David Deutsch and Richard Jozsa. Rapid solution of problems by quantum computation. 1992.

Implementation

- Mach-Zehnder interferometer implementation.
  https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/SinglePhotonLab/SinglePhotonLab.html

Mathematical description of the algorithm

$$|0\rangle \xrightarrow{H} |+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{cases} \xrightarrow{I} |+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} & \xrightarrow{H} |0\rangle \\ \xrightarrow{Z} |-\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} & \xrightarrow{H} |1\rangle \\ \xrightarrow{-Z} -|-\rangle = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} & \xrightarrow{H} -|1\rangle \\ \xrightarrow{-ZZ=-I} -|+\rangle = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} & \xrightarrow{H} -|0\rangle \end{cases}$$

Bloch sphere

Representation of pure states of a single qubit

$$|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle$$

- θ polar angle
- ϕ azimuthal angle

Euler's formula

$$e^{i\phi} = \cos\phi + i\sin\phi$$

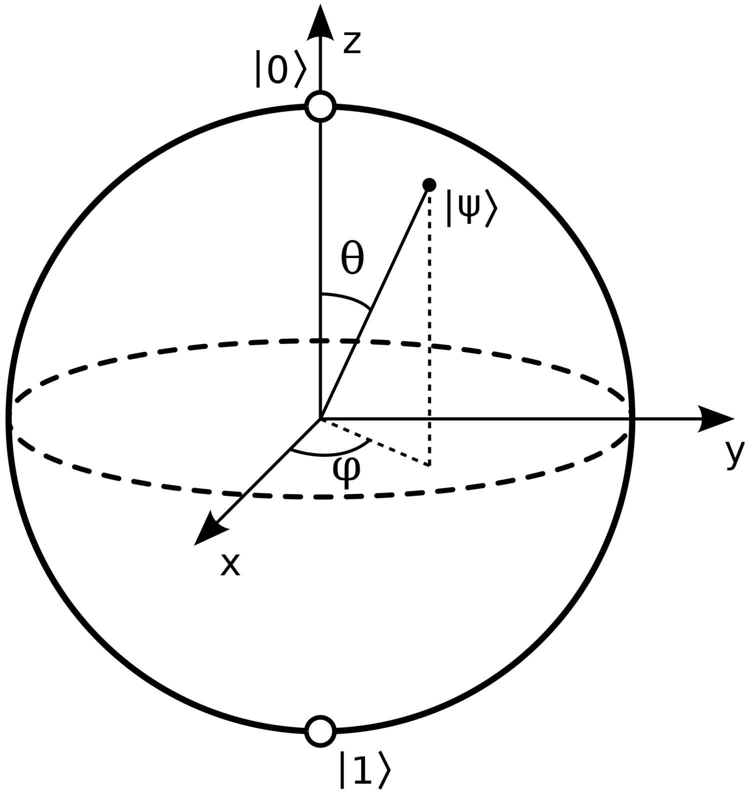

Figure: Source: Wikimedia

Bloch sphere

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

$$e^{i\phi} = \cos\phi + i\sin\phi$$

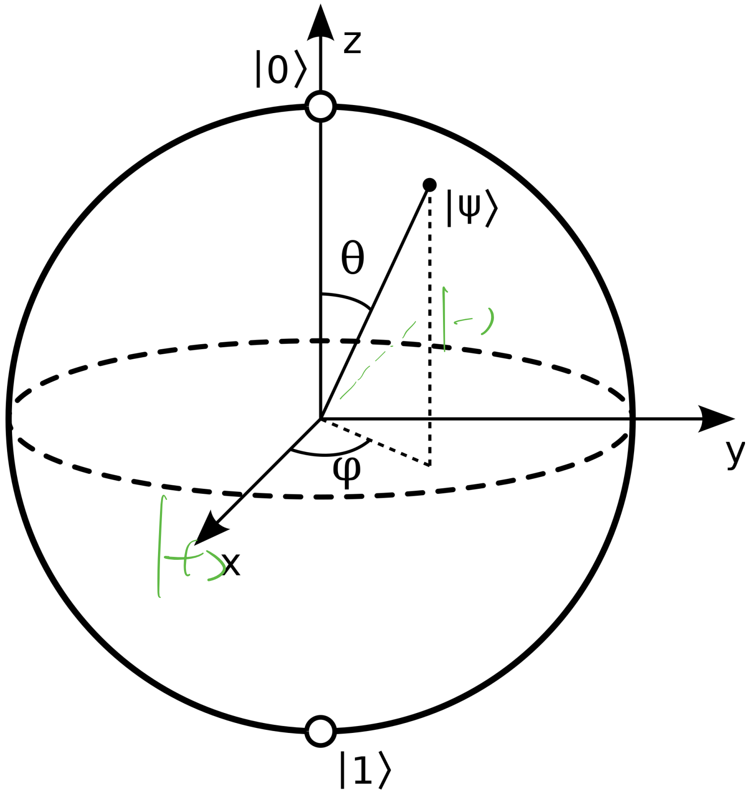

Figure: Source: Wikimedia

Important locations on the Bloch sphere

- $|+\rangle = H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
- $|-\rangle = H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Bloch sphere

$$|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle$$

$$e^{i\phi} = \cos\phi + i\sin\phi$$

Rotations around the Bloch sphere

- $$R_x(\theta) = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}X$$
- $$R_y(\theta) = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Y$$
- $$R_z(\theta) = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Z$$

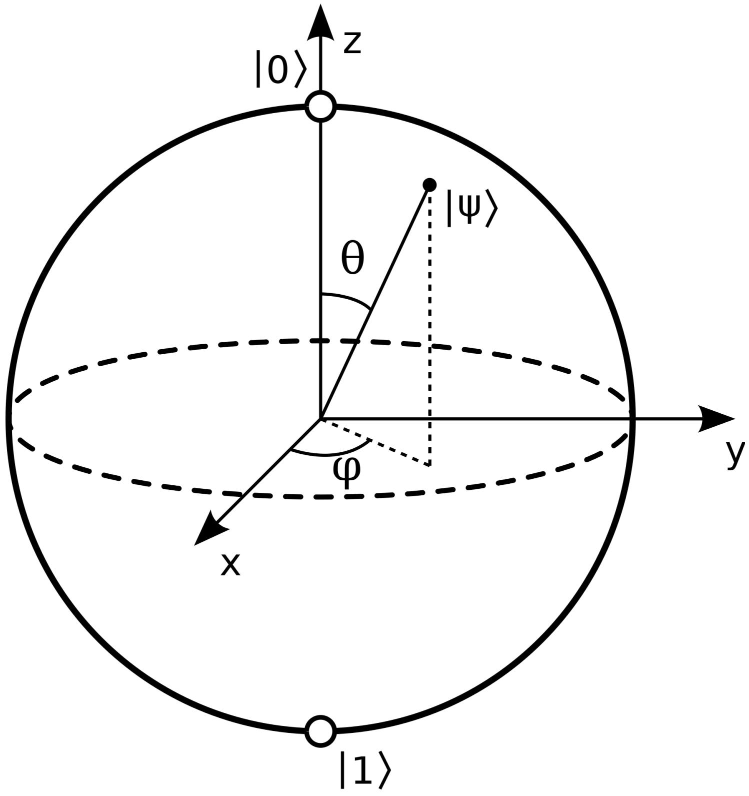

Figure: Source: Wikimedia

Exercise: The Qubit's Gambit

- Assume that the Earth is spherical.
- Let the North Pole be $|0\rangle$.
- Let $|+\rangle$ be on the prime meridian.
- What is the qubit state represented by the geographical location of Old Queens?

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$$b = 0.9999|0\rangle + 0.0001|1\rangle$$

Pauli Matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

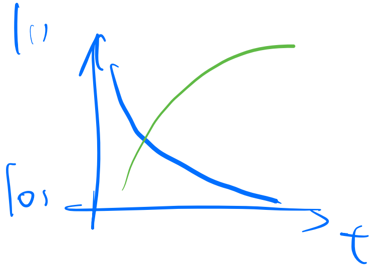
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$U = \alpha I + \beta X + \gamma Y + \delta Z$$

Hardware noise



1. Decoherence error
2. Gate error (imprecise control of single qubit, two qubit gates)
3. Measurement error

Technology	Coherence Time (s)	1-Qubit Gate Latency (s)	2-Qubit Gate Latency (s)	1-Qubit Gate Fidelity (%)	2-Qubit Gate Fidelity (%)	Mobile
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Solid State Nuclear spin	0.6 [183]	1.12e-4 [184] - 1.5e-4 [183]	1.2e-4 [185]*	99.6 - [184] - 99.95 [183]	89 [186] - 96 [185]*	NO
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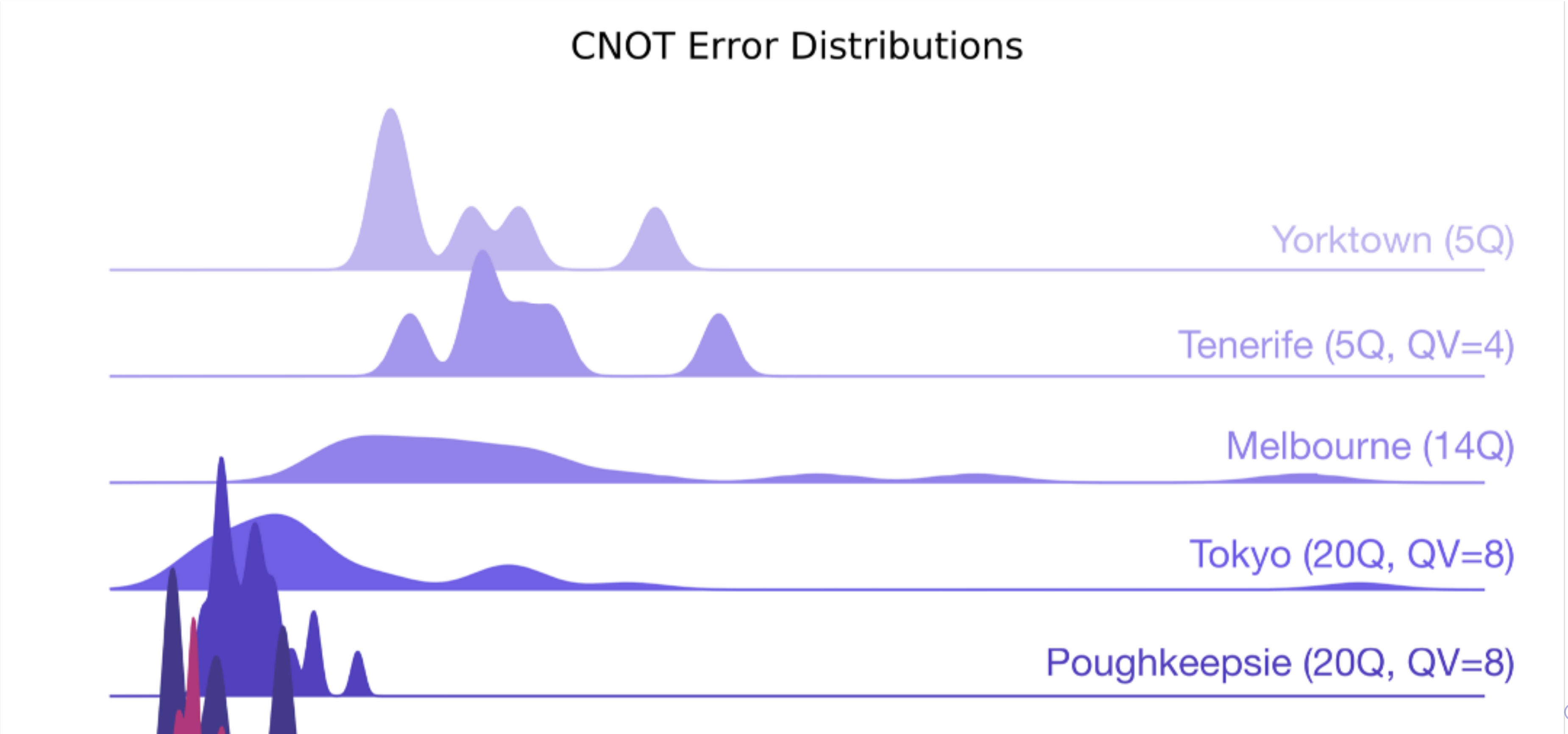
NMR	16.7 [158]	2.5e-4 [158] - 1e-3 [24]	2.7e-3 [158] - 1.0e-2 [24]	98.74 [24] - 99.60 [158]	98.23 [24] - 98.77 [158]	NO

Table 1. Metrics for various quantum technologies. * Nuclear/Electron Hybrid

Figure: Credit: [Resch and Karpuzcu, 2019]

Hardware noise

1. Decoherence error
2. Gate error (imprecise control of single qubit, two qubit gates)
3. Measurement error



Hardware noise

Stochastic, uncorrelated noise.

	Quantum noise mixtures (Pauli errors)	Quantum noise channels
Pauli-X type	Bit flip noise	Amplitude damping noise (related to T1 time)
Pauli-Z type	Phase flip noise	Phase damping noise (related to T2 time)
Combinations	Symmetric / asymmetric depolarizing noise	Generalized amplitude damping
Simulation technique	Can model as probabilistic ensembles of state vectors	Requires density matrix representation

Table: Summary of canonical quantum noise models.

$$0.64|+\rangle\langle+| + 0.36|-\rangle\langle-|$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= |1\rangle\langle 0| + |0\rangle\langle 1|$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}$$

Bit flip noise channel

$$|0\rangle \rightarrow BitFlip(0.64) \rightarrow \begin{cases} P(|0\rangle) = 0.64 \\ P(|1\rangle) = 0.36 \end{cases}$$

We represent such a mixture of quantum states as a density matrix:

$$0.64\,|0\rangle\langle 0| + 0.36\,|1\rangle\langle 1|$$
$$= 0.64 \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} + 0.36 \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}$$
$$= 0.64 \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 0.36 \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$
$$= \begin{bmatrix}0.64 & 0\\0 & 0.36\end{bmatrix} = \rho$$

(Conventions from [Nielsen and Chuang, 2011, Chapter 8.3])

Density matrix representation

$$0.64 |0\rangle \langle 0| + 0.36 |1\rangle \langle 1| = \begin{bmatrix} 0.64 & 0 \\ 0 & 0.36 \end{bmatrix}$$

More general representation:
$\rho = \sum_j p_j |\psi_j\rangle \langle \psi_j|$

$\sum_j p_j = 1$

Quantum (noise) channel

A quantum channel $\mathcal{E}(\rho)$ acts on mixed state ρ:

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger \quad = \quad \sum_k \left(E_k |\psi\rangle \right) \left(\langle\psi| E_k^\dagger \right)$$

$\rho = |\psi\rangle\langle\psi|$

Kraus Operators

Bit flip noise channel

The bit flip channel flips the state of a qubit with probability $1 - p$. It has two elements:

$$E_0 = \sqrt{p}I = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_1 = \sqrt{1-p}X = \sqrt{p}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

↓ $\sqrt{1-p}$

Bit flip noise channel

The bit flip noise channel $\mathcal{E}_{bitflip}(0.64)$ acts on the $|0\rangle$ state like so:

$$\mathcal{E}_{bitflip}(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix})$$

$$= \sum_k E_k \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} E_k^\dagger$$

$$= 0.8 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} 0.8 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0.6 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} 0.6 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.64 & 0 \\ 0 & 0.36 \end{bmatrix}$$

Bloch Vector

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2} = \begin{bmatrix} 0.64 & \\ & 0.36 \end{bmatrix}$$

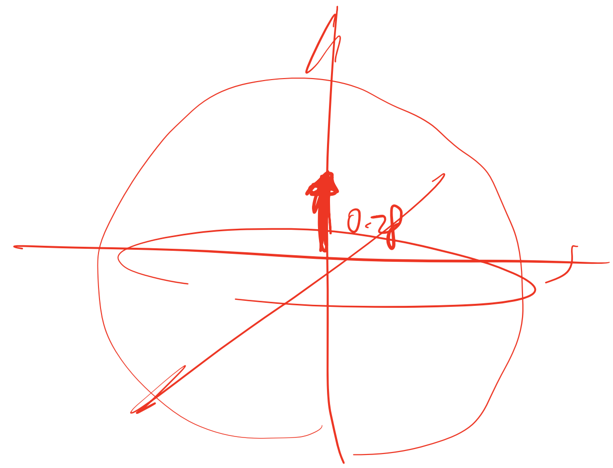
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} & -i \\ i & \end{bmatrix}$$

$$I + \vec{r}\cdot\vec{\sigma} = \begin{bmatrix} 1.28 & \\ & 0.72 \end{bmatrix}$$

$$\vec{r}\cdot\vec{\sigma} = \begin{bmatrix} 0.28 & \\ & -0.28 \end{bmatrix}$$

$$= 0.28\, Z$$

$$\vec{r} = \begin{bmatrix} x = 0 \\ y = 0 \\ z = 0.28 \end{bmatrix}$$



Phase flip noise channel

The phase flip channel flips the phase of a qubit with probability $1 - p$. It has two elements:

$$E_0 = \sqrt{p}I = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_1 = \sqrt{1-p}Z = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

~~$|+i\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ +i \end{bmatrix}$~~

$$\langle +i | +i \rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ +i \end{bmatrix}$$

$$= \frac{1}{2}\left(1 \cdot 1 + (-i)\cdot(+i) \right)$$

$$= \frac{1}{2}\left(1 + 1 \right)$$

$$= 1$$

$$Z|+i\rangle$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ +i \end{bmatrix}$$

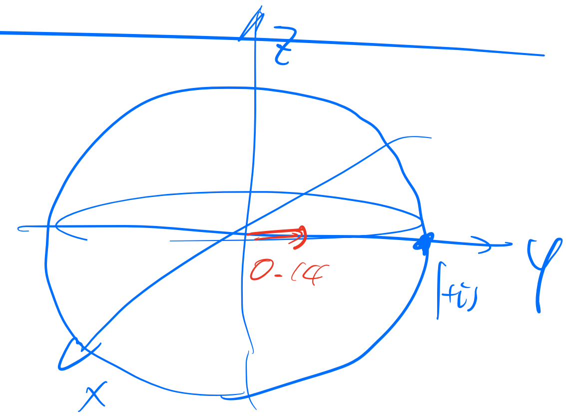
$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} = |-i\rangle$$

$$|-i\rangle\langle -i|$$

$$= \begin{bmatrix} 1 \\ -i \end{bmatrix} \begin{bmatrix} 1 & +i \end{bmatrix}$$

$$= \begin{bmatrix} 1 & +i \\ -i & 1 \end{bmatrix}$$

$$\mathcal{E}_{phaseflip(0.64)} \quad \text{on} \quad |+i\rangle$$



$$|+i\rangle = \begin{bmatrix} 1 \\ +i \end{bmatrix}$$

$$\rho = |+i\rangle\langle +i|$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ +i \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -i \\ +i & 1 \end{bmatrix}$$

$$\mathcal{E}_{phaseflip}\left(\frac{1}{2}\begin{bmatrix} 1 & -i \\ +i & 1 \end{bmatrix}\right)$$

$$= \sum_k E_k \frac{1}{2}\begin{bmatrix} 1 & -i \\ +i & 1 \end{bmatrix} E_k^{\dagger}$$

$$= 0.8\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} |+i\rangle\langle +i| \, 0.8 \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} + 0.6\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} |+i\rangle\langle +i| \, 0.6 \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$$

$$= 0.64 \cdot \frac{1}{2}\begin{bmatrix} 1 & -i \\ +i & 1 \end{bmatrix} + 0.36 \cdot \frac{1}{2}\begin{bmatrix} 1 & +i \\ -i & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -0.14i \\ +0.14i & 1/2 \end{bmatrix}$$

$$\rho = \begin{bmatrix} 1/2 & -0.14i \\ 0.14i & 1/2 \end{bmatrix}$$

$$\rho = \frac{I + \vec{r}\,\vec{\sigma}}{2} = \frac{I}{2} + \frac{r_x}{2}X + \frac{r_y}{2}Y + \frac{r_z}{2}Z$$

$$\begin{bmatrix} 0 & -0.14i \\ 0.14i & 0 \end{bmatrix} = \frac{r_x}{2}X + \frac{r_y}{2}Y + \frac{r_z}{2}Z$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\vec{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0.14 \\ 0 \end{bmatrix}$$

Depolarizing noise

$$\Delta_\lambda(\rho) = \sum_{i=0}^{3} K_i \rho K_i^\dagger$$

$$K_0 = \sqrt{1-\frac{3}{4}\lambda}\, I$$

$$K_1 = \sqrt{\frac{\lambda}{4}}\, X$$

$$K_2 = \sqrt{\frac{\lambda}{4}}\, Y$$

$$K_3 = \sqrt{\frac{\lambda}{4}}\, Z$$

total depol noise

$$\lambda = 1$$

The Maximally Mixed State

$$\rho = \frac{I}{2} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$= \frac{I + \vec{r}\cdot\vec{\sigma}}{2}$$

$$\vec{r} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{I}{2} = \begin{bmatrix} 1/2 & \\ & 1/2 \end{bmatrix} = P_{|0\rangle} |0\rangle\langle 0| + P_{|1\rangle} |1\rangle\langle 1|$$

$$P_{|0\rangle} = P_{|1\rangle} = 1/2$$

$$\frac{I}{2} = \begin{bmatrix} 1/2 & \\ & 1/2 \end{bmatrix} = P_{|+\rangle} |+\rangle + P_{|-\rangle} |-\rangle$$

$$= \frac{1}{2} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & \\ & 1/2 \end{bmatrix}$$

Hardware noise

	Quantum noise mixtures (Pauli errors)	Quantum noise channels
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Combinations	Symmetric / asymmetric depolarizing noise	Generalized amplitude damping
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Table: Summary of canonical quantum noise models.

Amplitude damping noise channel

The amplitude damping channel leaves $|0\rangle$ alone while probabilistically flipping $|1\rangle$. It has two elements:

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}$$
$$E_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$$

γ represents probability that $|1\rangle$ decays to $|0\rangle$

Hardware noise

1. Decoherence error
2. Gate error (imprecise control of single qubit, two qubit gates)
3. Measurement error

Technology	Coherence Time (s)	1-Qubit Gate Latency (s)	2-Qubit Gate Latency (s)	1-Qubit Gate Fidelity (%)	2-Qubit Gate Fidelity (%)	Mobile
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Figure: Credit: [Resch and Karpuzcu, 2019]

But what about correlated noise events?

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Quantum postulate 2: Composite systems

The state space of composite systems is the tensor product of state space of component systems.

Multiple qubits: the tensor product

Tensor product (also known as Kronecker product) of state vectors

$$|+\rangle \otimes |-\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \\ \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2} \\ \frac{1}{2} \\ \frac{-1}{2} \end{bmatrix} = \tfrac{1}{2}|00\rangle - \tfrac{1}{2}|01\rangle + \tfrac{1}{2}|10\rangle - \tfrac{1}{2}|11\rangle$$

Multiple qubits: the tensor product

Tensor product of unitary matrices

$$X \otimes I\left(\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} =$$

$$\begin{bmatrix} 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Circuit diagram representation:

Multiple qubits: the tensor product

Tensor product of state vectors

$$X\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes I|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Circuit diagram representation:

Multiple qubits: the tensor product

Exercise: proof by induction about the Hadamard transform

Show that $|+\rangle^{\otimes n} = \frac{1}{2^{n/2}} \sum_{m=0}^{2^n-1} |m\rangle$

Exercise: Superposition of computational basis states

1. Show that
$$(H \otimes H)\,|00\rangle = |+\rangle \otimes |+\rangle$$
2. Show that
$$|+\rangle^{\otimes n} = \frac{1}{2^{n/2}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{2^n \times 1}$$
3. Draw a quantum circuit that yields the above state for n qubits.

Entangled states: Bell state circuit

Bell state circuit

$$|00\rangle \xrightarrow{H \otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |\Phi^+\rangle$$

Can $|\Phi^+\rangle$ be treated as the tensor product (composition) of two individual qubits?

Prove that the Bell state cannot be factored into two single-qubit states

Bell state circuit

$$|00\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |\Phi^+\rangle$$

Can $|\Phi^+\rangle$ be treated as the tensor product (composition) of two individual qubits?
No.

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Bell states form an orthogonal basis set

1. $|00\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) = |\Phi^+\rangle$
2. $|01\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|01\rangle + |11\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|01\rangle + |10\rangle\Big) = |\Psi^+\rangle$
3. $|10\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle - |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle - |11\rangle\Big) = |\Phi^-\rangle$
4. $|11\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|01\rangle - |11\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|01\rangle - |10\rangle\Big) = |\Psi^-\rangle$

No-cloning theorem

There is no way to duplicate an arbitrary quantum state

Suppose a cloning operation U_c exists. Then:

- $$U_c(|\phi\rangle \otimes |\omega\rangle) = |\phi\rangle \otimes |\phi\rangle ,$$
  $$U_c(|\psi\rangle \otimes |\omega\rangle) = |\psi\rangle \otimes |\psi\rangle ,$$
  for arbitrary states $|\phi\rangle , |\psi\rangle$ we wish to copy.
- The overlap of the initial states is:
  $$\langle\phi| \otimes \langle\omega| \, |\psi\rangle \otimes |\omega\rangle = \langle\phi| \, |\psi\rangle \cdot \langle\omega| \, |\omega\rangle = \langle\phi| \, |\psi\rangle$$

No-cloning theorem

There is no way to duplicate an arbitrary quantum state

Suppose a cloning operation U_c exists. Then:

- $$U_c(|\phi\rangle \otimes |\omega\rangle) = |\phi\rangle \otimes |\phi\rangle \,,$$
  $$U_c(|\psi\rangle \otimes |\omega\rangle) = |\psi\rangle \otimes |\psi\rangle \,,$$
  for arbitrary states $|\phi\rangle \,, |\psi\rangle$ we wish to copy.
- The overlap of the final states is:
  $$\langle\phi| \otimes \langle\phi| \, |\psi\rangle \otimes |\psi\rangle = \langle\phi| \, |\psi\rangle \cdot \langle\phi| \, |\psi\rangle = (\langle\phi| \, |\psi\rangle)^2$$
- The overlap of the final states is also:
  $$\langle\phi| \otimes \langle\phi| \, |\psi\rangle \otimes |\psi\rangle = \langle\phi| \otimes \langle\omega| \, U^\dagger U \, |\psi\rangle \otimes |\omega\rangle = \langle\phi| \otimes \langle\omega| \, |\psi\rangle \otimes |\omega\rangle = \langle\phi| \, |\psi\rangle$$
- $(\langle\phi| \, |\psi\rangle)^2 = \langle\phi| \, |\psi\rangle$, so $\langle\phi| \, |\psi\rangle = 0$, or $\langle\phi| \, |\psi\rangle = 1$, $|\phi\rangle$ and $|\psi\rangle$ cannot be arbitrary states as claimed.

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Quantum postulate 3: Evolution

The time evolution of a state follows the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle$$

- Comes from the conservation of total energy in the closed system, one of the observables from the system state.
- Itself reflects a time-invariance.

$$\frac{\partial}{\partial t} |\psi(t)\rangle = \frac{-iH}{\hbar} |\psi(t)\rangle$$

$$|\psi(t)\rangle = e^{\frac{-iH}{\hbar}} |\psi(t)\rangle$$

Quantum postulate 3: Evolution

The evolution of a closed quantum system is a unitary transformation.

$$|\psi(t = t_1)\rangle = U\,|\psi(t = t_0)\rangle$$

- $|\psi_1\rangle = U\,|\psi_0\rangle$
- In a closed quantum system, $\langle\psi_1|\,|\psi_1\rangle = \langle\psi_0|\,U^\dagger U\,|\psi_0\rangle = \langle\psi_0|\,|\psi_0\rangle = 1$
- $U^\dagger U = I$, $U^\dagger = U^{-1}$; Such matrices U are unitary

Quantum postulate 3: Evolution

From unitary transformations we can show Hamiltonians in closed quantum systems must be hermitian

- $U|\psi\rangle = e^{\frac{-iH}{\hbar}}|\psi\rangle$
- $U^\dagger|\psi\rangle = e^{\frac{-(iH)^\dagger}{\hbar}}|\psi\rangle$
- $U^\dagger|\psi\rangle = U^{-1}|\psi\rangle = e^{\frac{iH}{\hbar}}|\psi\rangle$
- $(iH)^\dagger = -iH$, $A = iH$; such matrices A are called anti-Hermitian a.k.a. skew-Hermitian
- If iH is skew-Hermitian, H is Hermitian a.k.a. self-adjoint: $H^\dagger = H$

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