Quantum communications protocols: No cloning, quantum key distribution, dense coding, teleportation, stabilizer formalism, Bell's inequality

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Announcements

Tuesday Feb 17 will be last day for upvotes to influence midterm questions

Friday Feb 20 Midterm I.

Thursday Feb 26 4pm Rutgers QEC seminar

No live class Tues March 3. Watch the seminar recording instead.

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No-cloning theorem

There is no way to duplicate an arbitrary quantum state

Suppose a cloning operation U_c exists. Then:

- $$U_c(|\phi\rangle \otimes |\omega\rangle) = |\phi\rangle \otimes |\phi\rangle\,,$$
$$U_c(|\psi\rangle \otimes |\omega\rangle) = |\psi\rangle \otimes |\psi\rangle\,,$$
for arbitrary states $|\phi\rangle\,, |\psi\rangle$ we wish to copy.
- The overlap of the initial states is:

$$\langle\phi| \otimes \langle\omega|\,|\psi\rangle \otimes |\omega\rangle = \langle\phi|\,|\psi\rangle \cdot \langle\omega|\,|\omega\rangle = \langle\phi|\,|\psi\rangle$$

No-cloning theorem

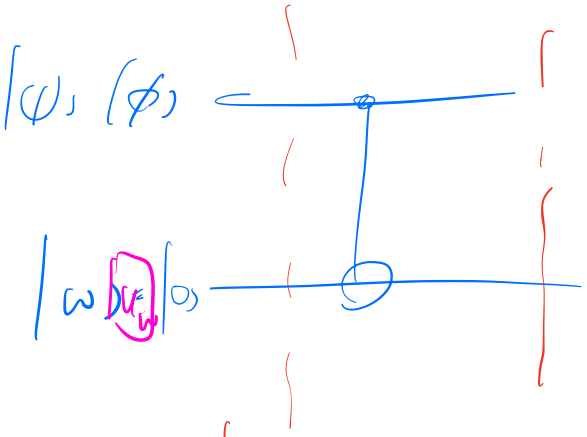
There is no way to duplicate an arbitrary quantum state

Suppose a cloning operation U_c exists. Then:

- $$U_c(|\phi\rangle \otimes |\omega\rangle) = |\phi\rangle \otimes |\phi\rangle\,,$$
  $$U_c(|\psi\rangle \otimes |\omega\rangle) = |\psi\rangle \otimes |\psi\rangle\,,$$
  for arbitrary states $|\phi\rangle\,, |\psi\rangle$ we wish to copy.
- The overlap of the final states is:
  $$\langle\phi| \otimes \langle\phi| \, |\psi\rangle \otimes |\psi\rangle = \langle\phi| \, |\psi\rangle \cdot \langle\phi| \, |\psi\rangle = (\langle\phi| \, |\psi\rangle)^2$$
- The overlap of the final states is also:
  $$\langle\phi| \otimes \langle\phi| \, |\psi\rangle \otimes |\psi\rangle = \langle\phi| \otimes \langle\omega| \, U^\dagger U \, |\psi\rangle \otimes |\omega\rangle = \langle\phi| \otimes \langle\omega| \, |\psi\rangle \otimes |\omega\rangle = \langle\phi| \, |\psi\rangle$$
- $(\langle\phi| \, |\psi\rangle)^2 = \langle\phi| \, |\psi\rangle$, so $\langle\phi| \, |\psi\rangle = 0$, or $\langle\phi| \, |\psi\rangle = 1$, $|\phi\rangle$ and $|\psi\rangle$ cannot be arbitrary states as claimed.

$|\psi\rangle = |0\rangle$ $\quad U_C|0\rangle \otimes |\omega\rangle = |0\rangle \otimes |0\rangle$

$|\phi\rangle = |1\rangle$ $\quad U_C|1\rangle \otimes |\omega\rangle = |1\rangle \otimes |1\rangle$



$|\psi\rangle = |0\rangle \otimes |0\rangle \rightarrow |0\rangle \otimes |0\rangle$

$|\phi\rangle = |1\rangle \otimes |0\rangle \rightarrow |1\rangle \otimes |1\rangle$

Exercise multi ver x1

$|\phi\rangle = |+i\rangle$

$|\psi\rangle = |-i\rangle$

U_C ?

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multi x2

$|\phi\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$

$|\psi\rangle = \dfrac{|01\rangle + |10\rangle}{\sqrt{2}}$

U_C

BB84 quantum key distribution

Charles Bennett and Gilles Brassard in 1984

- Using superposition, Alice and Bob can share a classical secret string
- Transmission of a quantum state cannot be intercepted due to the no-cloning theorem
- Alice and Bob can estimate the channel capacity
- Any attempt by Eve to eavesdrop will result in a detectable decrease in channel capacity
- `https://acephysics.net/tutorials/quantum-cryptography`

$$\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

Exercise

$$\begin{cases} |0\rangle \otimes |+\rangle \\ |0\rangle \otimes |-\rangle \\ |1\rangle \otimes |+\rangle \\ |1\rangle \otimes |-\rangle \end{cases}$$

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Entangled states: Bell state circuit

Bell state circuit

$$|00\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = |\Phi^+\rangle$$

Can $|\Phi^+\rangle$ be treated as the tensor product (composition) of two individual qubits?

Prove that the Bell state cannot be factored into two single-qubit states

Bell state circuit

$$|00\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = |\Phi^+\rangle$$

Can $|\Phi^+\rangle$ be treated as the tensor product (composition) of two individual qubits?
No.

Proof by contradiction:

- Suppose $|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \Big(\alpha|0\rangle + \beta|1\rangle\Big) \otimes \Big(\gamma|0\rangle + \delta|1\rangle\Big) = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$.
- $\alpha\delta = 0$, so either $\alpha = 0$ or $\delta = 0$.
- But $\alpha\gamma = \frac{1}{\sqrt{2}}$, so $\alpha \neq 0$.
- And $\beta\delta = \frac{1}{\sqrt{2}}$, so $\delta \neq 0$ too.

Bell states form an orthogonal basis set

1. $|00\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) = |\Phi^+\rangle$
2. $|01\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|01\rangle + |11\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|01\rangle + |10\rangle\Big) = |\Psi^+\rangle$
3. $|10\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle - |10\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle - |11\rangle\Big) = |\Phi^-\rangle$
4. $|11\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(|01\rangle - |11\rangle\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|01\rangle - |10\rangle\Big) = |\Psi^-\rangle$

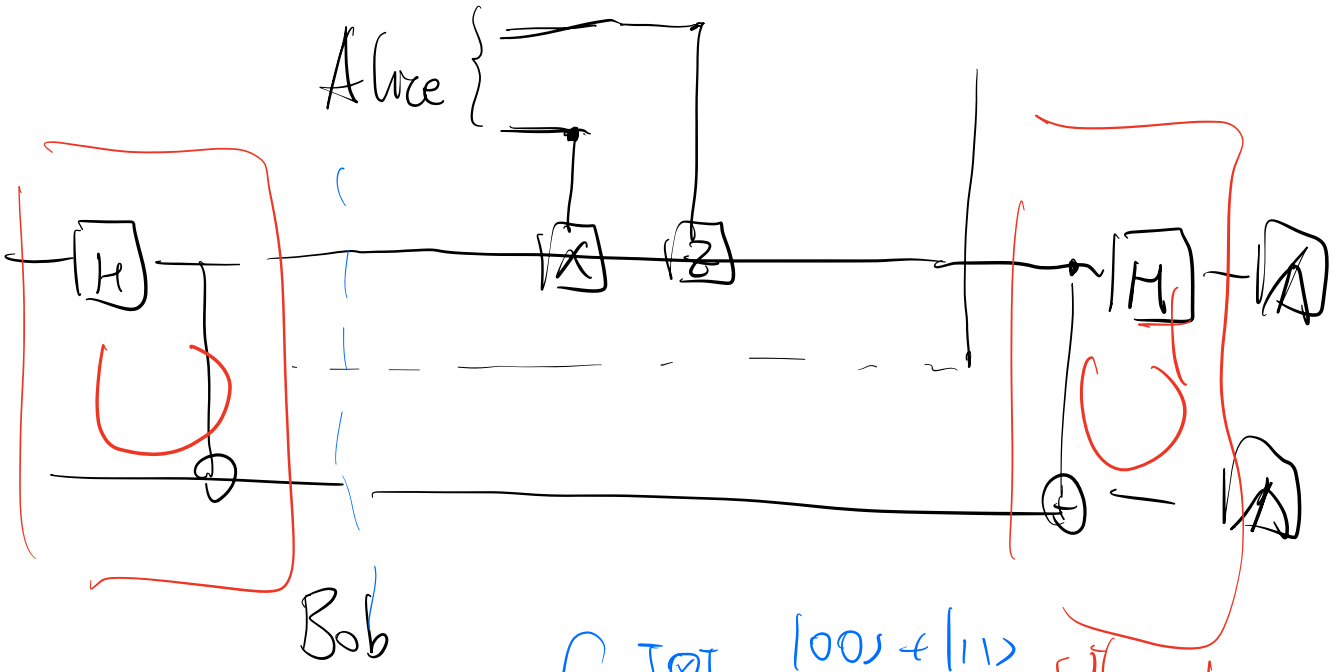
Superdense coding

Transmit 2 bits of classical information by sending 1 qubit

1. Alice wishes to tell Bob two bits of information: 00, 01, 10, or 11.
2. Alice and Bob each have one qubit of a Bell pair in state $|P\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big)$.
3. Alice performs I, X, Z, or ZX on her qubit; she then sends her qubit to Bob.
4. Bob measures in the Bell basis to receive 00, 01, 10, or 11.

Superdense coding circuit

`https://github.com/quantumlib/Cirq/blob/master/examples/superdense_coding.py`



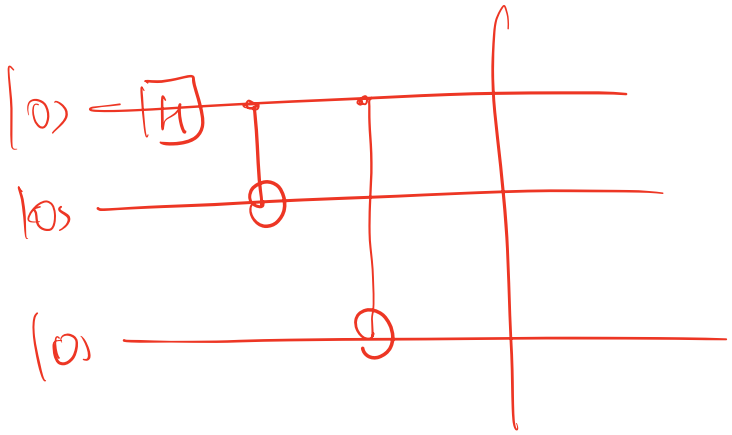
$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} \xrightarrow{I \otimes I} \frac{|00\rangle + |11\rangle}{\sqrt{2}} \xrightarrow{U} |00\rangle$$

$$\frac{|10\rangle + |01\rangle}{\sqrt{2}} \xrightarrow{I \otimes X} \frac{|10\rangle + |01\rangle}{\sqrt{2}} \xrightarrow{U} |01\rangle$$

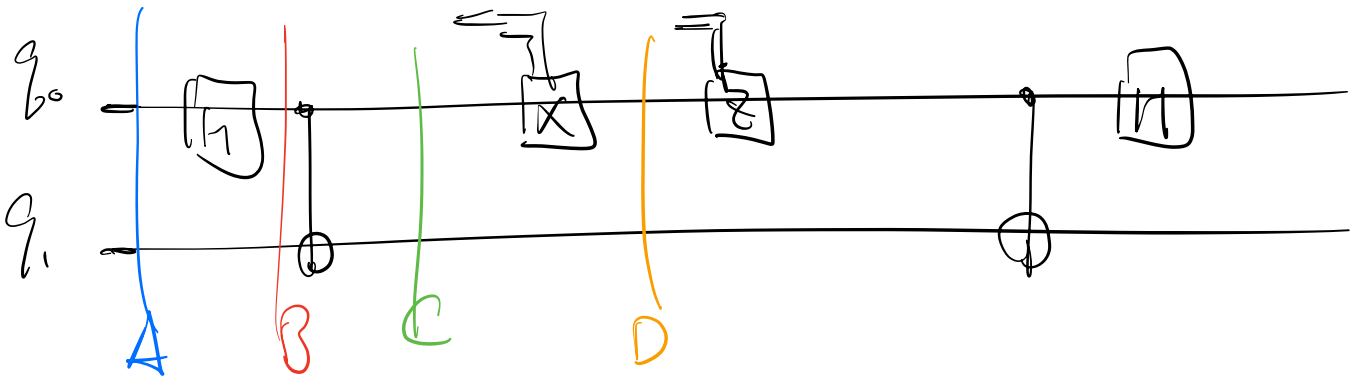
$$\frac{|00\rangle - |11\rangle}{\sqrt{2}} \xrightarrow{I \otimes Z} \frac{|00\rangle - |11\rangle}{\sqrt{2}} \xrightarrow{U} |10\rangle$$

$$\frac{|10\rangle - |01\rangle}{\sqrt{2}} \xrightarrow{-Y \otimes I} \frac{|10\rangle - |01\rangle}{\sqrt{2}} \xrightarrow{U} |11\rangle$$

$$\frac{|a\rangle|x\rangle + |1\rangle|x \oplus 1\rangle}{\sqrt{2}}$$



$$\frac{|000\rangle + |111\rangle}{\sqrt{2}}$$ GHZ



A

Z		X		r
q_0	q_1	q_0	q_1	
1	0	0	0	0
0	1	0	0	0

ZI
IZ
$\Big\}\ |00\rangle$

$ZI\,|00\rangle = |00\rangle$

$IZ\,|00\rangle = |00\rangle$

$H\otimes I$

$(Z\otimes I)(I\otimes Z)$

$= Z\otimes Z$

$(I\otimes Z)(Z\otimes I)$

$= I\otimes I$

$(Z\otimes Z)(Z\otimes Z) =$

$(Z\otimes I)(Z\otimes I) =$

$H|Z|0\rangle\langle 0|Z^\dagger H^\dagger =$

$HZH = X$

B

Z		X		r
q_0	q_1	q_0	q_1	
0	0	1	0	0
0	1	0	0	0

XI
IX
$\Big\}\ |+0\rangle$

$(X\otimes I)(X\otimes I)$

$= I\otimes I$

$(X\otimes I)(I\otimes X)$

$= X\otimes X$

$X\otimes Z\,|+0\rangle$

$= X|+\rangle\otimes Z|0\rangle$

$= |+\rangle\otimes|0\rangle$

C

$\mathrm{CNOT}\,(X \otimes I)\,\mathrm{CNOT}^{\dagger}$

$$= \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix} \left[\; \right] \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix}$$

$= XX$

$\mathrm{CNOT}\,(I \otimes Z)\,\mathrm{CNOT}^{\dagger}$

$= ZZ$

C	q_1 q_2 (Z)	q_1 q_2 (X)	v
	1 1	0 0	0
	0 0	1 1	0

$\left.\begin{matrix} XX \\ ZZ \end{matrix}\right\} \; |\Phi^{+}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$

$(X \otimes X)(X \otimes X)$

$= I \otimes I$

$(X \otimes X)(Z \otimes Z)$

$= -YY$

$-YY \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$

$\rho = |\Phi^{+}\rangle\langle\Phi^{+}|$

$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix}1&0&0&1\end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{bmatrix}$

$= \dfrac{II + XX - YY + ZZ}{2}$

$= \dfrac{\begin{bmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{bmatrix} + \begin{bmatrix}&&&1\\&&1&\\&1&&\\1&&&\end{bmatrix} + \begin{bmatrix}&&&1\\&&-1&\\&-1&&\\1&&&\end{bmatrix} + \begin{bmatrix}1&&&\\&-1&&\\&&-1&\\&&&1\end{bmatrix}}{2}$

$= \rho = |\Phi^{+}\rangle\langle\Phi^{+}|$

$XXX^\dagger = IX \cdot X$

$XZX^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -Z$

D $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left| \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \right.$

$\left.\begin{matrix} -ZZ \\ XX \end{matrix}\right\}$ ( )

$I \otimes I$

$(-Z \otimes Z)(X \otimes X)$

$= +YY$

Superdense coding

Transmit 2 bits of classical information by sending 1 qubit

1. Alice wishes to tell Bob two bits of information: 00, 01, 10, or 11.
2. Alice and Bob each have one qubit of a Bell pair in state
   $|P\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big)$.
3. Alice performs I, X, Z, or ZX on her qubit; she then sends her qubit to Bob.
4. Bob measures in the Bell basis to receive 00, 01, 10, or 11.

Alice applies different operators on her qubit so Bob measures the message

1. $|P\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) \xrightarrow{I\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle + |10\rangle \Big) \xrightarrow{H\otimes I} |00\rangle$
2. $|P\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) \xrightarrow{X\otimes I} \frac{1}{\sqrt{2}}\Big(|10\rangle + |01\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|11\rangle + |01\rangle \Big) \xrightarrow{H\otimes I} |01\rangle$
3. $|P\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) \xrightarrow{Z\otimes I} \frac{1}{\sqrt{2}}\Big(|00\rangle - |11\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(|00\rangle - |10\rangle \Big) \xrightarrow{H\otimes I} |10\rangle$
4. $|P\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) \xrightarrow{ZX\otimes I} \frac{1}{\sqrt{2}}\Big(-|10\rangle + |01\rangle \Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(-|11\rangle + |01\rangle \Big) \xrightarrow{H\otimes I} |11\rangle$

What is it that gives quantum computers an advantage compared to classical computing?

- Superposition?
- Entanglement?
- Both?
- Neither?

Importance of representations in quantum intuition, programming, and simulation

- Conventional quantum circuits and state vector view of QC conceals symmetries, hinders intuition.
- Classical simulation of quantum computing is actually tractable for *a certain subset* of quantum gates.
- Both the logical and native gatesets in a quantum architecture need to be *universal* for quantum advantage.

Several views/representations of quantum computing

- Programming has several views: functional programming, procedural programming.
- Physics has several views: Newtonian, Lagrangian, Hamiltonian

Different views reveal different symmetries, offer different intuition.

Several views/representations of quantum computing

- Schrödinger: state vectors and density matrices
- Heisenberg: stabilizer formalism
- Tensor-network
- Feynman: path sums

A survey of these representations of quantum computing is given in Chapter 9 of this recent book [Ding and Chong, 2020].

Several views/representations of quantum computing

- Schrödinger: state vectors and density matrices
- Heisenberg: stabilizer formalism
- Tensor-network
- Feynman: path sums
- Binary decision diagrams (new?)
- Logical satisfiability equations

Schrödinger view

- In Schrödinger quantum mechanics description, emphasis on how *states* evolve.
- $CNOT_{0,1}(H_0 \otimes I_1)\left|00\right\rangle = CNOT_{0,1}\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} =$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

- The Schrodinger view requires exponential storage: A quantum computer with N qubits can be in superposition of 2^N basis states, requires 2^N amplitudes to fully specify state.

Heisenberg view / stabilizer formalism

- In Heisenberg quantum mechanics description, emphasis on how *operators* evolve.
- If we limit operations to the Clifford gates (a subset of quantum gates), simulation tractable in polynomial time and space.
- Covers some quantum algorithms: quantum superdense coding, quantum teleportation, Deutsch-Jozsa, Bernstein-Vazirani, quantum error correction, most quantum error correction protocols.
- A model for probabilistic (but not quantum) computation.

Concrete example on Bell state circuit

$$CNOT_{0,1}(H_0 \otimes I_1)\,|00\rangle$$

1. Start with N qubits with initial state $|0\rangle^{\otimes N}$.
2. Represent the state as its group of stabilizers—$|00\rangle : \{IZ, ZI\}$
3. When simulating the quantum circuit, decompose the Clifford gates to stabilizer gates $\{CNOT, H, P\}$.
4. Apply each of the stabilizer gates to the stabilizer representation.

- Hadamard on first qubit—$|+\rangle\,|0\rangle$: $\{IZ, XI\}$
- CNOT on both qubits—$\frac{|00\rangle+|11\rangle}{\sqrt{2}}$: $\{ZZ, XX\}$

Representing a state as its group of stabilizers

- A unitary operator U stabilizes a pure state $|\psi\rangle$ if $U|\psi\rangle = |\psi\rangle$

Representing a state as its group of stabilizers

1. I stabilizes everything.
2. $-I$ stabilizes nothing.
3. X stabilizes $|+\rangle$: $X|+\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |+\rangle$
4. $-X$ stabilizes $|-\rangle$: $-X|-\rangle = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = |-\rangle$
5. Y stabilizes $|+i\rangle$: $Y|+i\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = |+i\rangle$
6. $-Y$ stabilizes $|-i\rangle$: $-Y|-i\rangle = -\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix} = |-i\rangle$
7. Z stabilizes $|0\rangle$: $Z|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$
8. $-Z$ stabilizes $|1\rangle$: $-Z|1\rangle = -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$

Representing a state as its group of stabilizers

In other words,

1. $|0\rangle$ is stabilized by $\{I, Z\}$
2. $|1\rangle$ is stabilized by $\{I, -Z\}$
3. $|+\rangle$ is stabilized by $\{I, X\}$
4. $|-\rangle$ is stabilized by $\{I, -X\}$
5. $|+i\rangle$ is stabilized by $\{I, Y\}$
6. $|-i\rangle$ is stabilized by $\{I, -Y\}$

Special places on the Bloch sphere

$$
\begin{aligned}
|\psi\rangle &= \alpha\,|0\rangle + \beta\,|1\rangle \\
&= |\alpha|[cos(\gamma) + i \cdot sin(\gamma)]\,|0\rangle \\
&+ |\beta|[cos(\gamma+\phi) + i \cdot sin(\gamma+\phi)]\,|1\rangle \\
&= cos(\frac{\theta}{2})e^{i\gamma}\,|0\rangle + sin(\frac{\theta}{2})e^{i(\gamma+\phi)}\,|1\rangle
\end{aligned}
$$

Enforces $|\alpha|^2 + |\beta|^2 = 1$

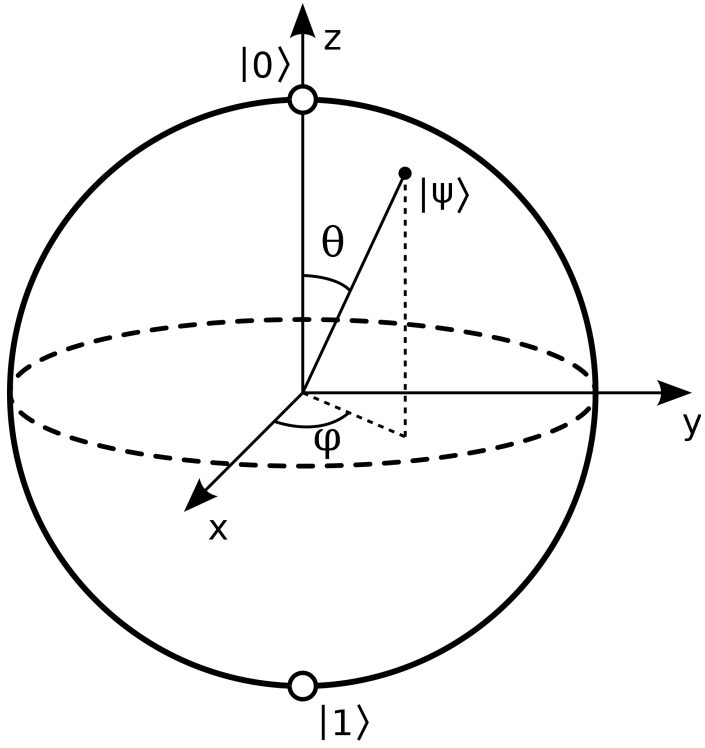


Figure: Bloch sphere showing pole states.
Source: Wikimedia.

Special places on the Bloch sphere

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Enforces $|\alpha|^2 + |\beta|^2 = 1$

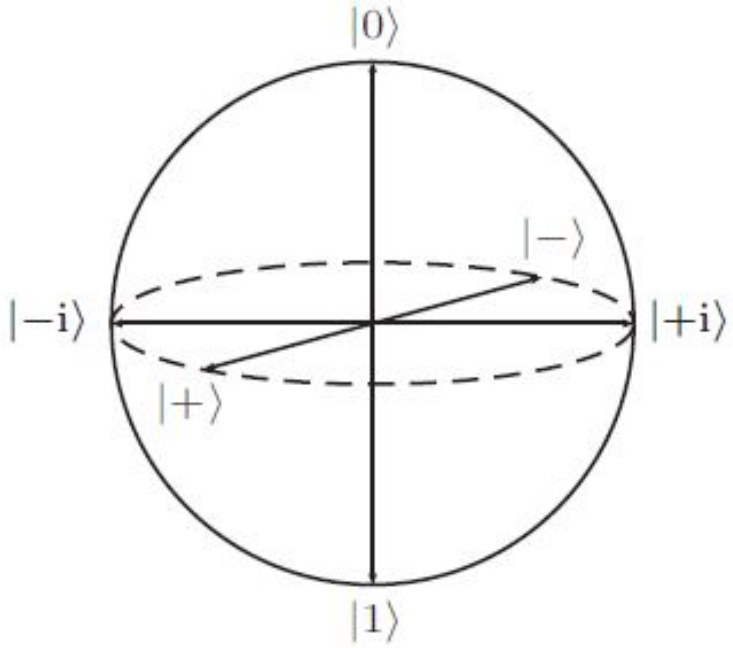

Figure: Bloch sphere showing pole states.
Source: Wikimedia.

Representing a state as its group of stabilizers

For multi-qubit states, the group of stabilizers is the cartesian product of the single-qubit stabilizers

- $|00\rangle = |0\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, Z \otimes I, Z \otimes Z\}$
- $\frac{|00\rangle+|10\rangle}{\sqrt{2}} = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes |0\rangle = |+\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, X \otimes I, X \otimes Z\}$
- $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ is stabilized by $\{I \otimes I, X \otimes X, -Y \otimes Y, Z \otimes Z\}$

Representing a state as its group of stabilizers

- Critical result from group theory: for any N-qubit stabilized state, only N elements needed to specify group—a result from abstract algebra group theory [Nielsen and Chuang, 2002, Appendix 2]
- So long as the quantum circuit consists only of Clifford gates, only N elements needed to specify whole quantum state.
- Contrast against 2^N amplitudes needed to specify a general N-qubit quantum state vector.
- For example a two-qubit states needs four amplitues $\{a_0, a_1, a_2, a_3\}$ to specify quantum state $|\psi\rangle = a_0 |00\rangle + a_1 |01\rangle + a_2 |10\rangle + a_3 |11\rangle$.

Representing a state as its group of stabilizers

Critical result from group theory: for any N-qubit stabilized state, only N elements needed to specify group.

1. $|0\rangle$ is stabilized by $\{I, Z\}$, Z is generator
2. $|1\rangle$ is stabilized by $\{I, -Z\}$, -Z is generator
3. $|+\rangle$ is stabilized by $\{I, X\}$, X is generator
4. $|-\rangle$ is stabilized by $\{I, -X\}$, -X is generator
5. $|+i\rangle$ is stabilized by $\{I, Y\}$, Y is generator
6. $|-i\rangle$ is stabilized by $\{I, -Y\}$, -Y is generator
7. $|0\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, Z \otimes I, Z \otimes Z\}$, $\{I \otimes Z, Z \otimes I\}$ is generator
8. $|+\rangle \otimes |0\rangle$ is stabilized by $\{I \otimes I, I \otimes Z, X \otimes I, X \otimes Z\}$, $\{I \otimes Z, X \otimes I\}$ is generator
9. $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ is stabilized by $\{I \otimes I, X \otimes X, -Y \otimes Y, Z \otimes Z\}$, $\{X \otimes X, Z \otimes Z\}$ is generator

Concrete example on Bell state circuit

$$CNOT_{0,1}(H_0 \otimes I_1)\,|00\rangle$$

1. Start with N qubits with initial state $|0\rangle^{\otimes N}$.
2. Represent the state as its group of stabilizers—$|00\rangle : \{IZ, ZI\}$
3. When simulating the quantum circuit, decompose the Clifford gates to stabilizer gates $\{CNOT, H, P\}$.
4. Apply each of the stabilizer gates to the stabilizer representation.

- Hadamard on first qubit—$|+\rangle\,|0\rangle$: $\{IZ, XI\}$
- CNOT on both qubits—$\frac{|00\rangle+|11\rangle}{\sqrt{2}}$: $\{ZZ, XX\}$

Stabilizer gates: $\{CNOT, H, P\}$

1. Hadamard gate: induces superpositions.
2. CNOT gate: induces entanglement.
3. Phase gate: induces complex phases. $P = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

- Despite featuring superposition, entanglement, and complex amplitudes, is *not* universal for quantum computing.
- We shall see that the deeply symmetrical structure of these gates prevent access to full quantum Hilbert space.

Stabilizer gates are a generator for Pauli gates (i.e., Clifford gates decompose to stabilizer gates)

Pauli gates are rotations around respective axes by π.

- $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = PP$
- $X = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = HZH$
- $Y = iXZ$
- $X^2 = Y^2 = Z^2 = I$
- Symmetry is similar to quaternions.
- With Clifford gates consisting of $\{CNOT, H, P, I, X, Y, Z\}$, sufficient to build many quantum algorithms, including: quantum superdense coding, quantum teleportation, Deutsch-Jozsa, Bernstein-Vazirani, quantum error correction, most quantum error correction protocols.

Single qubit stabilizer gates bounce stabilizer states around an octahedron on the Bloch sphere

$$P|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

$$P|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = i|1\rangle$$

$$P|+\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = |+i\rangle$$

$$P|-\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix} = |-i\rangle$$

$$P|+i\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = |-\rangle$$

$$P|-i\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |+\rangle$$

Single qubit stabilizer gates bounce stabilizer states around an octahedron on the Bloch sphere

$$H|0\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |+\rangle$$

$$H|1\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = |-\rangle$$

$$H|+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

$$H|-\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

$$H|+i\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix} = |-i\rangle$$

$$H|-i\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = |+i\rangle$$

Apply each of the stabilizer gates to the stabilizer representation.

- Phase:
  1. $Z \to Z$
  2. $-Z \to -Z$
  3. $X \to Y$
  4. $-X \to -Y$
  5. $Y \to -X$
  6. $-Y \to X$
- Hadamard:
  1. $Z \to X$
  2. $-Z \to -X$
  3. $X \to Z$
  4. $-X \to -Z$
  5. $Y \to -Y$
  6. $-Y \to Y$

Single qubit stabilizer gates bounce stabilizer states around an octahedron on the Bloch sphere

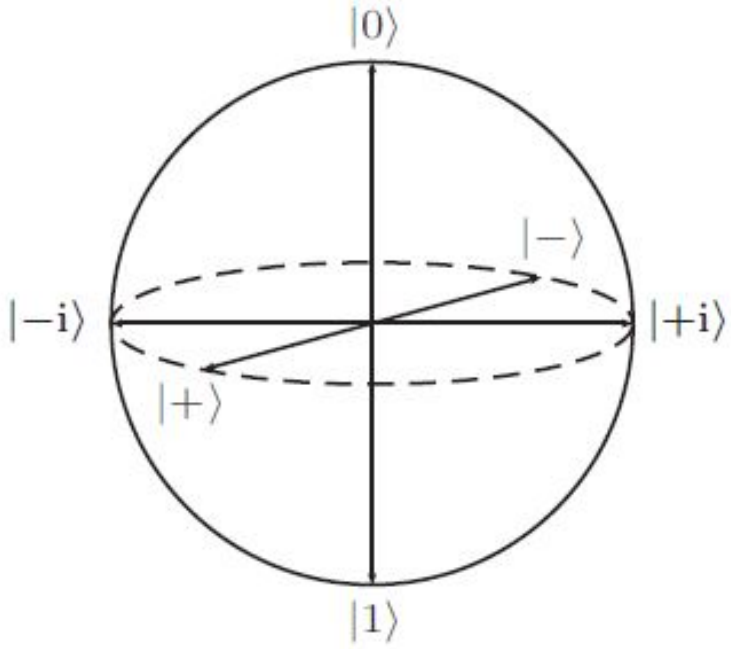

Figure: Bloch sphere showing pole states. Source: Wikimedia.

Apply each of the stabilizer gates to the stabilizer representation.

- CNOT:
  1. $X \otimes I \rightarrow X \otimes X$
  2. $I \otimes X \rightarrow I \otimes X$
  3. $Z \otimes I \rightarrow Z \otimes I$
  4. $I \otimes Z \rightarrow Z \otimes Z$

Concrete example on Bell state circuit

$$CNOT_{0,1}(H_0 \otimes I_1)\,|00\rangle$$

1. Start with N qubits with initial state $|0\rangle^{\otimes N}$.
2. Represent the state as its group of stabilizers—$|00\rangle : \{IZ, ZI\}$
3. When simulating the quantum circuit, decompose the Clifford gates to stabilizer gates $\{CNOT, H, P\}$.
4. Apply each of the stabilizer gates to the stabilizer representation.

- Hadamard on first qubit—$|+\rangle\,|0\rangle$: $\{IZ, XI\}$
- CNOT on both qubits—$\frac{|00\rangle+|11\rangle}{\sqrt{2}}$: $\{ZZ, XX\}$

Gottesman-Knill theorem and its implications

- Gottesman-Knill theorem states that there exists a classical algorithm that simuates any stabilizer circuit in polynomial time.
- Any quantum state created by a Clifford circuit, even if it has lots of superpositions and entanglement, is easy to classically simulate.
- Quantum computers need at least one non-Clifford gate to achieve universal quantum computation.
- The T gate, where $TT = P, PP = Z$ is one common choice.
- There are results showing that a quantum circuit is only exponentially hard to simulate w.r.t. the number of T-gates.

Exercises

1. Show that the T gate is non-Clifford.
2. Show that the controlled-S gate is non-Clifford.
3. Show that the Toffoli gate is non-Clifford.

References

- Main sources: [Gottesman, 1998] [Aaronson,]
- Further reference on separation of probabilistic and quantum computing: [Van Den Nes, 2010]
- Further reference on applications in classical simulation of Clifford quantum circuits: [Aaronson and Gottesman, 2004]
- Further reference on applications in classical simulation of general quantum circuits: [Bravyi and Gosset, 2016]

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Quantum teleportation

"Teleport" a qubit state by transmitting classical information

1. Alice wishes to give Bob a qubit state $|Q\rangle$.
2. Alice and Bob each have one qubit of a Bell pair in state $|P\rangle$.
3. Alice first entangles $|Q\rangle$ and $|P\rangle$; then, she measures her local two qubits.
4. Alice tells Bob (via classical means) her two-bit measurement result.
5. Bob uses Alice's two bits to perform I, X, Z, or ZX on his qubit to obtain $|Q\rangle$.

Depending on if Alice measures 00, 01, 10, or 11, Bob applies I, X, Z, or ZX to recover $|Q\rangle$

$$
\begin{aligned}
|Q\rangle \otimes |P\rangle &= \big(\alpha|0\rangle + \beta|1\rangle\big) \otimes \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big) \\
&= \frac{\alpha}{\sqrt{2}}|0\rangle\big(|00\rangle + |11\rangle\big) + \frac{\beta}{\sqrt{2}}|1\rangle\big(|00\rangle + |11\rangle\big) \\
&\xrightarrow{CNOT_{0,1}} \frac{\alpha}{\sqrt{2}}|0\rangle\big(|00\rangle + |11\rangle\big) + \frac{\beta}{\sqrt{2}}|1\rangle\big(|10\rangle + |01\rangle\big) \\
&\xrightarrow{H\otimes I\otimes I} \frac{\alpha}{2}\big(|0\rangle + |1\rangle\big)\big(|00\rangle + |11\rangle\big) + \frac{\beta}{2}\big(|0\rangle - |1\rangle\big)\big(|10\rangle + |01\rangle\big) \\
&= \frac{1}{2}|00\rangle\big(\alpha|0\rangle + \beta|1\rangle\big) \text{ Alice measures 00 so Bob applies I} \\
&+ \frac{1}{2}|01\rangle\big(\alpha|1\rangle + \beta|0\rangle\big) \text{ Alice measures 01 so Bob applies X} \\
&+ \frac{1}{2}|10\rangle\big(\alpha|0\rangle - \beta|1\rangle\big) \text{ Alice measures 10 so Bob applies Z} \\
&+ \frac{1}{2}|11\rangle\big(\alpha|1\rangle - \beta|0\rangle\big) \text{ Alice measures 11 so Bob applies ZX}
\end{aligned}
$$

$$\alpha|0\rangle\otimes(|00\rangle+|11\rangle) + \beta|1\rangle\otimes(|00\rangle+|11\rangle)$$

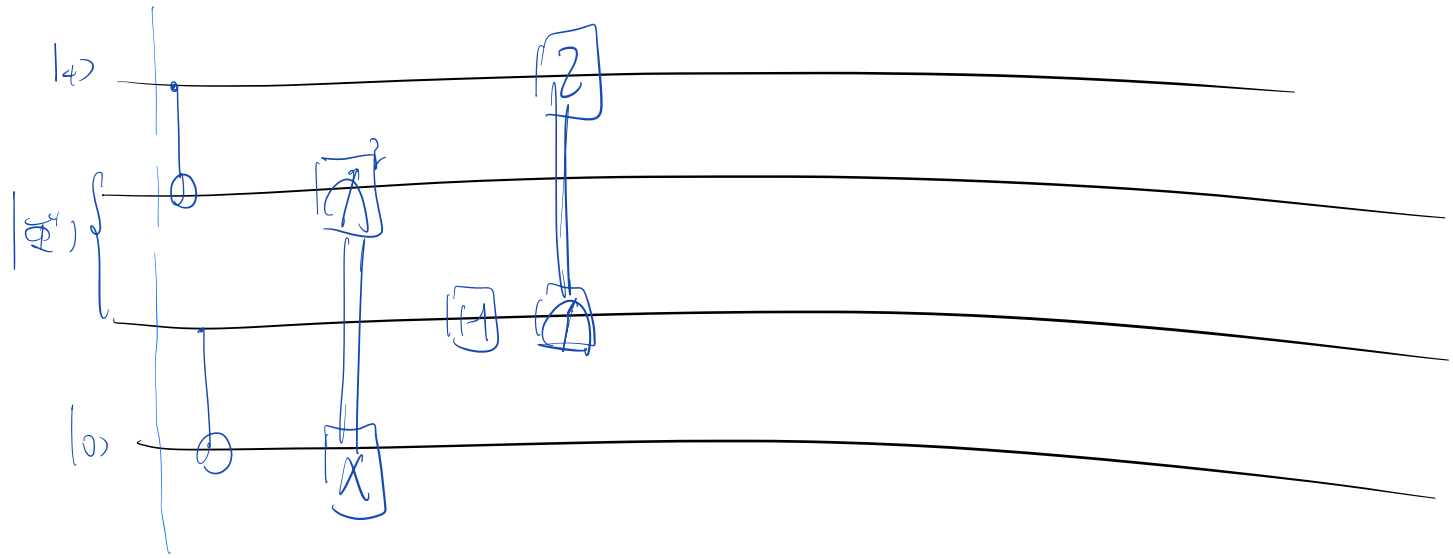
$$\alpha|0\rangle\otimes(|11\rangle+|00\rangle) + \beta|1\rangle\otimes(|10\rangle+|01\rangle)$$

$$\alpha(|11\rangle+|00\rangle)\otimes|0\rangle + \beta(|10\rangle+|01\rangle)\otimes|1\rangle$$

$$\alpha(|01\rangle+|10\rangle)\otimes|-\rangle + \beta(|11\rangle+|00\rangle)\otimes|+\rangle$$

$$= \alpha|000\rangle+\alpha|011\rangle+\alpha|100\rangle+\alpha|111\rangle + \beta|001\rangle-\beta|010\rangle-\beta|101\rangle+\beta|110\rangle$$

$$= |11\rangle(\alpha|0\rangle+\beta|1\rangle) + |00\rangle(\alpha|0\rangle+\beta|1\rangle) + |10\rangle(\alpha|1\rangle-\beta|0\rangle) + |01\rangle(\alpha|1\rangle-\beta|0\rangle)$$



$$|+\rangle \otimes |\psi\rangle + |0\rangle$$

$$= \left(|0\rangle+|1\rangle\right) + \left(\left(|00\rangle+|11\rangle\right) + |0\rangle\right)$$

$$= |0000\rangle + |0110\rangle + |0001\rangle + |0111\rangle$$

↓ Cnot

$$|0000\rangle + |0110\rangle + |0001\rangle + |0111\rangle$$

$$|000\rangle + |111\rangle \qquad |010\rangle + |101\rangle$$

↓H ↓H

$$|000\rangle + |000\rangle \qquad |010\rangle - |010\rangle$$

$$= |000\rangle + |010\rangle + |101\rangle + |111\rangle$$

$$|111\rangle + |000\rangle \qquad |101\rangle - |010\rangle$$

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EPR paradox

- When quantum physics was first discovered, the mathematics of entanglement led to shocking conclusions.
- If you can keep systems coherent (isolated), they can exhibit superposition and entanglement.
- Einstein and others: there shouldn't be "spooky action at a distance" so there must be some local hidden-variable. The task was then to prove or disprove local hidden-variables.
- But protocols and experiments like Hardy's, GHZ, CHSH, and Aspect experimentally rejected local hidden-variable theory.

CHSH game: Test of entanglement

Two isolated parties Alice and Bob

- Alice gets coin toss x, replies a
- Bob gets coin toss y, replies b

Goal: maximize $a \oplus b = x \wedge y$

x	y	$x \wedge y$	$a \oplus b$	winning options for (a, b)
0	0	0	0	(0,0) or (1,1)
0	1	0	0	(0,0) or (1,1)
1	0	0	0	(0,0) or (1,1)
1	1	1	1	(0,1) or (1,0)

Best classical strategy to maximize $a \oplus b = x \wedge y$

Proof that any assignment to a and b cannot always satisfy $a \oplus b = x \wedge y$

1. Let a_0 be Alice's response if she sees $x = 0$
2. Let a_1 be Alice's response if she sees $x = 1$
3. Let b_0 be Bob's response if she sees $y = 0$
4. Let b_1 be Bob's response if she sees $y = 1$

Satisfy $a \oplus b = x \wedge y$

1. $a_0 \oplus b_0 = 0$
2. $a_0 \oplus b_1 = 0$
3. $a_1 \oplus b_0 = 0$
4. $a_1 \oplus b_1 = 1$

Sum (mod 2) of left side

$$(a_0 \oplus b_0) \oplus (a_0 \oplus b_1) \oplus (a_1 \oplus b_0) \oplus (a_1 \oplus b_1) =$$
$$(a_0 \oplus a_0) \oplus (a_1 \oplus a_1) \oplus (b_0 \oplus b_0) \oplus (b_1 \oplus b_1) = 0$$

Sum (mod 2) of right side

1

Best classical strategy to maximize $a \oplus b = x \wedge y$

- Even if the two shared randomness, the random coin toss of x and y prevents use of shared randomness.
- Best you can do is 3/4.
- Give a couple ways of getting 3/4

A quantum strategy to maximize $a \oplus b = x \wedge y$

Alice and Bob share entangled pair $|\Phi\rangle$

$$|\Phi\rangle = \frac{1}{\sqrt{12}}\Big(3\,|00\rangle + |01\rangle + |10\rangle - |11\rangle\Big)$$

$(x, y) = (0, 0)$ So Alice and Bob both apply I:

$$(I \otimes I)\,|\Phi\rangle = \frac{1}{\sqrt{12}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Measurement yields

$$\begin{cases} (a, b) = (0, 0), \text{ a win, with probability } \frac{9}{12} \\ (a, b) = (0, 1), \text{ a loss, with probability } \frac{1}{12} \\ (a, b) = (1, 0), \text{ a loss, with probability } \frac{1}{12} \\ (a, b) = (1, 1), \text{ a win, with probability } \frac{1}{12} \end{cases}$$

A quantum strategy to maximize $a \oplus b = x \wedge y$

Alice and Bob share entangled pair $|\Phi\rangle$

$$|\Phi\rangle = \frac{1}{\sqrt{12}}\Big(3\,|00\rangle + |01\rangle + |10\rangle - |11\rangle\Big)$$

$(x, y) = (0, 1)$ So Alice applies I, Bob applies H:

$$(I \otimes H)\,|\Phi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\end{bmatrix}\frac{1}{\sqrt{12}}\begin{bmatrix}3\\ 1\\ 1\\ -1\end{bmatrix} = \frac{1}{2\sqrt{6}}\begin{bmatrix}4\\ 2\\ 0\\ 2\end{bmatrix}$$

Measurement yields

$$\begin{cases}(a, b) = (0, 0), \text{ a win, with probability } \frac{4}{6}\\ (a, b) = (0, 1), \text{ a loss, with probability } \frac{1}{6}\\ (a, b) = (1, 0), \text{ a loss, with probability } 0\\ (a, b) = (1, 1), \text{ a win, with probability } \frac{1}{6}\end{cases}$$

A quantum strategy to maximize $a \oplus b = x \wedge y$

Alice and Bob share entangled pair $|\Phi\rangle$

$|\Phi\rangle = \frac{1}{\sqrt{12}}\left(3\,|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)$

$(x, y) = (1, 0)$ So Alice applies H, Bob applies I:

$$(H \otimes I)\,|\Phi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \frac{1}{\sqrt{12}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2\sqrt{6}} \begin{bmatrix} 4 \\ 0 \\ 2 \\ 2 \end{bmatrix}$$

Measurement yields

$$\begin{cases} (a, b) = (0, 0), \text{ a win, with probability } \frac{4}{6} \\ (a, b) = (0, 1), \text{ a loss, with probability } 0 \\ (a, b) = (1, 0), \text{ a loss, with probability } \frac{1}{6} \\ (a, b) = (1, 1), \text{ a win, with probability } \frac{1}{6} \end{cases}$$

A quantum strategy to maximize $a \oplus b = x \wedge y$

Alice and Bob share entangled pair $|\Phi\rangle$

$$|\Phi\rangle = \frac{1}{\sqrt{12}}\Big(3\,|00\rangle + |01\rangle + |10\rangle - |11\rangle\Big)$$

$(x, y) = (1, 1)$ So Alice and Bob both apply H:

$$(H \otimes H)\,|\Phi\rangle = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \frac{1}{\sqrt{12}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{4\sqrt{3}} \begin{bmatrix} 4 \\ 4 \\ 4 \\ 0 \end{bmatrix}$$

Measurement yields

$$\begin{cases} (a, b) = (0, 0), \text{ a loss, with probability } \frac{1}{3} \\ (a, b) = (0, 1), \text{ a win, with probability } \frac{1}{3} \\ (a, b) = (1, 0), \text{ a win, with probability } \frac{1}{3} \\ (a, b) = (1, 1), \text{ a loss, with probability } 0 \end{cases}$$

A quantum strategy to maximize $a \oplus b = x \wedge y$

Sum of winning chances?

Philosophical interpretations of quantum mechanics

Cannot have both locality and realism

- Locality: "means that information and causation act locally, not faster than light"
- Realism: "means that physical systems have definite, well-defined properties (even if those properties may be unknown to us)"

Source: de Wolf. Quantum Computing: Lecture Notes

Unpalatable choices

- Keep locality and sacrifice realism: no definite narrative of the world
- Keep realism and sacrifice locality: spooky-action-at-a-distance

References


Aaronson, S.
Lecture 28, tues may 2: Stabilizer formalism.


Aaronson, S. and Gottesman, D. (2004).
Improved simulation of stabilizer circuits.
Phys. Rev. A, 70:052328.


Bravyi, S. and Gosset, D. (2016).
Improved classical simulation of quantum circuits dominated by clifford gates.
Phys. Rev. Lett., 116:250501.


Ding, Y. and Chong, F. T. (2020).
Quantum computer systems: Research for noisy intermediate-scale quantum computers.
Synthesis Lectures on Computer Architecture, 15(2):1–227.


Gottesman, D. (1998).
The heisenberg representation of quantum computers.
arXiv preprint quant-ph/9807006.


Nielsen, M. A. and Chuang, I. (2002).
Quantum computation and quantum information.


Van Den Nes, M. (2010).
Classical simulation of quantum computation, the gottesman-knill theorem, and slightly beyond.
Quantum Info. Comput., 10(3):258–271.