

Basic quantum algorithms: Deutsch / Deutsch-Jozsa

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Semester Plan

Basic Algorithms

Q Error Correction

Everything you can do with 4 qubits

b/w
midterm 1
midterm 2

Today's Plan

Review of stab tableaux w.r.t. Bell's circuit measurement

Deutsch

↳ statevector

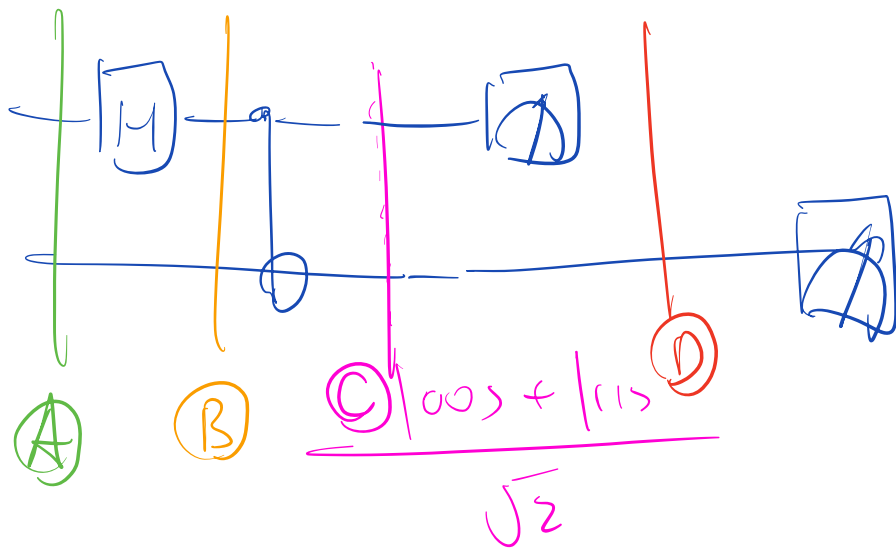
↳ stab view

D-J

HW Exercises

558 Projects

Bell State Stabilizer View



A

S_0, S_1, X_0, X_1, r

$S_0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 = Z \otimes I$

$S_1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 = I \otimes Z$

$$\begin{aligned} (Z \otimes I) |00\rangle &= |00\rangle \\ (I \otimes Z) |00\rangle &= |00\rangle \\ (Z \otimes Z) |00\rangle &= |00\rangle \\ (I \otimes I) |00\rangle &= |00\rangle \end{aligned}$$

③

	z_0	z_1	x_0	x_1	r	
S_0	0	0	1	0	0	XI
S_1	0	1	0	0	0	Iz

$$\text{Stab} \left\{ \begin{array}{c} XZI \\ ZIZ \\ XZI \\ IZI \end{array} \right\} \left| \begin{array}{c} +0S \\ +0S \\ +0S \\ +0S \end{array} \right\rangle = \left| \begin{array}{c} +0S \\ +0S \\ +0S \\ +0S \end{array} \right\rangle$$

$$\rho = \frac{XI + IZ + XZ + II}{2} = \left| \begin{array}{c} +0S \\ +0S \\ +0S \\ +0S \end{array} \right\rangle$$

④

	z_0	z_1	x_0	x_1	r	
S_0	0	0	1	1	0	XX
S_1	1	1	0	0	0	ZZ

$$\begin{aligned} \text{Cnot}(XI) \text{Cnot}^\dagger &= XX \\ \text{Cnot}(IZ) \text{Cnot}^\dagger &= ZZ \end{aligned}$$

$$\text{Stabs} \left\{ \begin{array}{c} XX \\ ZZ \\ -YY \\ II \end{array} \right\} \left| \begin{array}{c} \Phi \\ \Phi \\ \Phi \\ \Phi \end{array} \right\rangle = \left| \begin{array}{c} \Phi \\ \Phi \\ \Phi \\ \Phi \end{array} \right\rangle$$

$$\left| \bar{\Phi} \right\rangle \left\langle \bar{\Phi} \right| = \frac{II + XX - YY + ZZ}{2}$$

① Measure Z_0

	Z_0	Z_1	X_0	X_1	r	
S_0	1	0	0	0	0/1	$+ZI/-ZI$
S_1	1	1	0	0	0	ZI

top qubit meas "0"

ZI, ZI, II, II

$$\rho = \frac{II + IZ + ZI + ZZ}{4}$$

$$= \frac{(I+Z)}{2} \otimes \frac{(I+Z)}{2}$$

$$= |0\rangle\langle 0| \otimes |0\rangle\langle 0|$$

top qubit meas "1"

$-ZI, ZI, -IZ, II$

$$\rho = \frac{II - IZ - ZI + ZZ}{4}$$

$$= \frac{(I-Z)}{2} \otimes \frac{(I-Z)}{2}$$

$$= |1\rangle\langle 1| \otimes |1\rangle\langle 1|$$

Promise algorithms vs. unstructured search

'70s



Quantum algorithms offer exponential speedup in “promise” problems

A progression of related algorithms:

1. Deutsch's -
2. Deutsch-Jozsa
3. Bernstein-Vazirani -
4. Simon's
5. Shor's -

'93
↓
'96

↓
property of the f(x)

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Deutsch-Jozsa algorithm: extending Deutsch's algorithm to more qubits

- The state after applying oracle U

- Lemma: the Hadamard transform

- The state after the final set of Hadamards

- Probability of measuring upper register to get 0

Exercises

Deutsch-Jozsa algorithm: simplest quantum algorithm showing advantage vs. classical

A Heist

- ▶ You break into a bank vault. The bank vault has 2^n bars. Three possibilities: all are gold, half are gold and half are fake, or all are fake.
- ▶ Even if you steal just one gold bar, it is enough to fund your escape from the country, forever evading law enforcement.
- ▶ You do not want to risk stealing from a bank vault with only fake bars.
- ▶ You have access to an oracle $f(x)$ that tells you if gold bar x is real.
- ▶ Using the oracle sounds the alarm, so you only get to use it once.

Deutsch-Jozsa algorithm: simplest quantum algorithm showing advantage vs. classical

More formal description

► The 2^n bars are either fake or gold. $f : \{0, 1\}^n \rightarrow \{0, 1\}$.

► Three possibilities:

1. All are fake. f is constant. $f(x) = 0$ for all $x \in \{0, 1\}^n$.

2. All are gold. f is constant. $f(x) = 1$ for all $x \in \{0, 1\}^n$.

3. Half fake half gold. f is balanced.

$$\left| \{x \in \{0, 1\}^n : f(x) = 0\} \right| = \left| \{x \in \{0, 1\}^n : f(x) = 1\} \right| = 2^{n-1}$$

► The oracle U works as follows: $U |c\rangle |t\rangle = |c\rangle |t \oplus f(c)\rangle$

► Try deciding if f is constant or balanced using oracle U only once.

What is in the oracle

For $n = 1$, four possibilities

	f_0	f_1	f_2	f_3
$f(0)$	0	0	1	1
$f(1)$	0	1	0	1
	f is constant 0	f is balanced	f is balanced	f is constant 1

Demonstration of Deutsch-Jozsa for the $n = 1$ case

Output of circuit is $c = 0$ iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle \otimes |1\rangle = |0\rangle |1\rangle = |01\rangle$
2. After first set of Hadamards: $H \otimes H \left(|0\rangle \otimes |1\rangle \right) = H |0\rangle \otimes H |1\rangle = |+\rangle \otimes |-\rangle =$
$$\left(\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \right) \otimes \left(\frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \right) = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

From here, let's take an aside via matrix-vector multiplication to build intuition with interference and phase kickback.

Demonstration of Deutsch-Jozsa for the $n = 1$ case

Output of circuit is $c = 0$ iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle \otimes |1\rangle = |0\rangle |1\rangle = |01\rangle$

2. After first set of Hadamards: $H \otimes H \left(|0\rangle \otimes |1\rangle \right) = |+\rangle |-\rangle =$
 $\left(\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \right) \left(\frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \right) = \frac{1}{2} \left(|0\rangle (|0\rangle - |1\rangle) + |1\rangle (|0\rangle - |1\rangle) \right)$

3. After applying oracle U :

$$U \frac{1}{2} \left(|0\rangle (|0\rangle - |1\rangle) + |1\rangle (|0\rangle - |1\rangle) \right) = \frac{1}{2} \left(|0\rangle (|f(0) \oplus 0\rangle - |f(0) \oplus 1\rangle) + \right. \\ \left. |1\rangle (|f(1) \oplus 0\rangle - |f(1) \oplus 1\rangle) \right) = \frac{1}{2} \left(|0\rangle (|f(0)\rangle - |f(\bar{0})\rangle) + |1\rangle (|f(1)\rangle - |f(\bar{1})\rangle) \right)$$

Demonstration of Deutsch-Jozsa for the $n = 1$ case

Output of circuit is $c = 0$ iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle \otimes |1\rangle = |0\rangle |1\rangle = |01\rangle$
2. After first set of Hadamards: $\frac{1}{2} \left(|0\rangle (|0\rangle - |1\rangle) + |1\rangle (|0\rangle - |1\rangle) \right)$
3. After applying oracle U : $U \frac{1}{2} \left(|0\rangle (|0\rangle - |1\rangle) + |1\rangle (|0\rangle - |1\rangle) \right) =$
 $\frac{1}{2} \left(|0\rangle (|f(0)\rangle - |f(\bar{0})\rangle) + |1\rangle (|f(1)\rangle - |f(\bar{1})\rangle) \right)$
4. This last expression can be factored depending on f :
 $U \frac{1}{2} \left(|0\rangle (|0\rangle - |1\rangle) + |1\rangle (|0\rangle - |1\rangle) \right) =$
$$\begin{cases} \frac{1}{2} (|0\rangle + |1\rangle) (|f(0)\rangle - |f(\bar{0})\rangle) & \text{if } f(0) = f(1) \\ \frac{1}{2} (|0\rangle - |1\rangle) (|f(0)\rangle - |f(\bar{0})\rangle) & \text{if } f(0) \neq f(1) \end{cases} = \begin{cases} |+\rangle |-\rangle & \text{if } f(0) = f(1) \\ |-\rangle |-\rangle & \text{if } f(0) \neq f(1) \end{cases}$$

The trick where oracle's output on $|t\rangle$ affects phase of $|c\rangle$ is called phase kickback.

Demonstration of Deutsch-Jozsa for the $n = 1$ case

Output of circuit is $c = 0$ iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle \otimes |1\rangle = |0\rangle |1\rangle = |01\rangle$
2. After first set of Hadamards: $\frac{1}{2} \left(|0\rangle (|0\rangle - |1\rangle) + |1\rangle (|0\rangle - |1\rangle) \right)$
3. After applying oracle U :
$$U_{\frac{1}{2}} \left(|0\rangle (|0\rangle - |1\rangle) + |1\rangle (|0\rangle - |1\rangle) \right) = \begin{cases} |+\rangle |-\rangle & \text{if } f(0) = f(1) \\ |-\rangle |-\rangle & \text{if } f(0) \neq f(1) \end{cases}$$
4. After applying second H on top qubit:
$$\begin{cases} H \otimes I(|+\rangle |-\rangle) = |0\rangle |-\rangle & \text{if } f(0) = f(1) \\ H \otimes I(|-\rangle |-\rangle) = |1\rangle |-\rangle & \text{if } f(0) \neq f(1) \end{cases}$$

Deutsch-Jozsa programs and systems

Algorithm

David Deutsch and Richard Jozsa. Rapid solution of problems by quantum computation. 1992.

Programs

Google Cirq programming example.

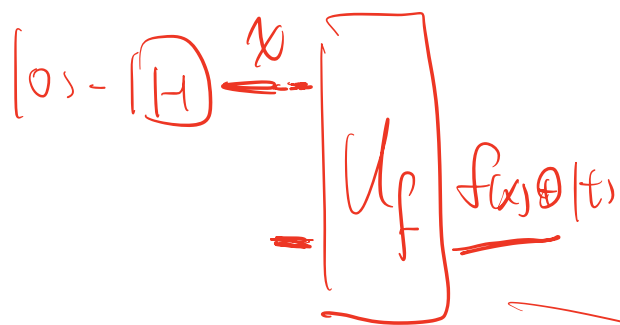
Implementation

- ▶ Mach-Zehnder interferometer implementation.
https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/SinglePhotonLab/SinglePhotonLab.html
- ▶ Ion trap implementation. Gulde et al. Implementation of the Deutsch-Jozsa algorithm on an ion-trap quantum computer. Letters to Nature. 2003.

Mach-Zehnder interferometer implementation of Deutsch's algorithm

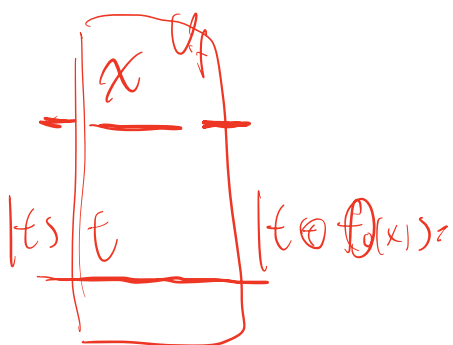
$$|0\rangle \xrightarrow{H} |+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \left\{ \begin{array}{ll} \xrightarrow{I} |+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} & \xrightarrow{H} |0\rangle \\ \xrightarrow{Z} |-\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} & \xrightarrow{H} |1\rangle \\ \xrightarrow{-Z} -|-\rangle = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} & \xrightarrow{H} -|1\rangle \\ \xrightarrow{-ZZ=-I} -|+\rangle = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} & \xrightarrow{H} -|0\rangle \end{array} \right.$$

Deutsch State Vector View



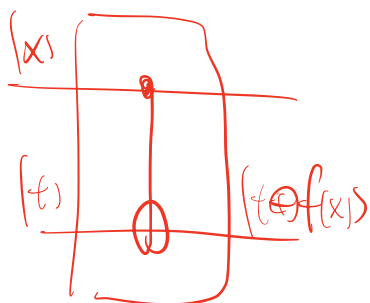
$$f_0(0) = 0$$

$$f_0(1) = 0$$



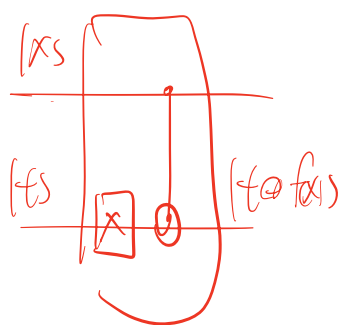
$$f_1(0) = 0$$

$$f_1(1) = 1$$



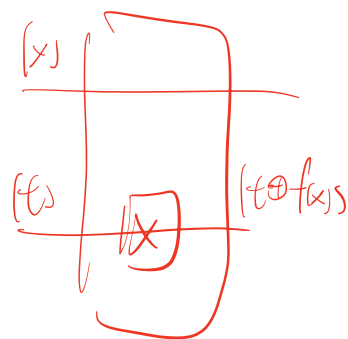
$$f_2(0) = 1$$

$$f_2(1) = 0$$



$$f_3(0) = 1$$

$$f_3(1) = 1$$



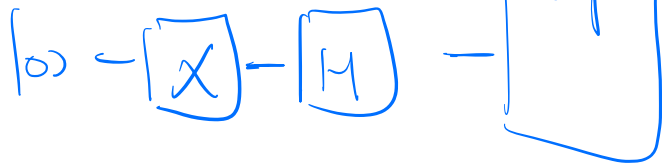


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- The state after applying oracle U

- Lemma: the Hadamard transform

- The state after the final set of Hadamards

- Probability of measuring upper register to get 0

Exercises

Deutsch-Jozsa algorithm: Deutsch's algorithm for the $n > 1$ case

The state after the first set of Hadamards

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle^{\otimes n} \otimes |1\rangle = |0 \dots 0\rangle |1\rangle = |0 \dots 01\rangle$
2. After first set of Hadamards: $|+\rangle^{\otimes n} \otimes |-\rangle = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} |c\rangle \otimes |-\rangle$

Deutsch's algorithm: Deutsch-Jozsa for the $n = 1$ case

The state after applying oracle U

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle^{\otimes n} \otimes |1\rangle = |0 \dots 0\rangle |1\rangle = |0 \dots 01\rangle$
2. After first set of Hadamards: $|+\rangle^{\otimes n} \otimes |-\rangle = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} |c\rangle \otimes |-\rangle$
3. After applying oracle U :

$$\begin{aligned} U\left(|+\rangle^{\otimes n} \otimes |-\rangle\right) &= \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} |c\rangle \otimes \left(\frac{|f(c)\rangle - |f(\bar{c})\rangle}{\sqrt{2}}\right) \\ &= \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \end{aligned}$$

Lemma: the Hadamard transform

$$H^{\otimes n} |c\rangle = \frac{1}{2^{n/2}} \sum_{m=0}^{2^n-1} (-1)^{c \cdot m} |m\rangle$$



$$\begin{aligned} H^{\otimes n} |c\rangle &= H |c_0\rangle \otimes H |c_1\rangle \otimes \dots \otimes H |c_{n-1}\rangle \\ &= \frac{1}{\sqrt{2}} \left(|0\rangle + (-1)^{c_0} |1\rangle \right) \otimes \frac{1}{\sqrt{2}} \left(|0\rangle + (-1)^{c_1} |1\rangle \right) \otimes \dots \otimes \frac{1}{\sqrt{2}} \left(|0\rangle + (-1)^{c_{n-1}} |1\rangle \right) \\ &= \frac{1}{2^{n/2}} \sum_{m=0}^{2^n-1} (-1)^{c_0 m_0 + c_1 m_1 + \dots + c_{n-1} m_{n-1} \bmod 2} |m\rangle \end{aligned}$$

► Try it out for $n = 1$: $H^{\otimes 1} |c\rangle = \frac{1}{2^{1/2}} \sum_{m=0}^{2^1-1} (-1)^{c \cdot m} |m\rangle =$

$$\frac{1}{\sqrt{2}} (-1)^0 |0\rangle + \frac{1}{\sqrt{2}} (-1)^c |1\rangle = \begin{cases} \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle = |+\rangle & \text{if } |c\rangle = |0\rangle \\ \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle = |-\rangle & \text{if } |c\rangle = |1\rangle \end{cases}$$

Deutsch-Jozsa algorithm: Deutsch's algorithm for the $n > 1$ case

The state after applying oracle U

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle^{\otimes n} \otimes |1\rangle = |0 \dots 0\rangle |1\rangle = |0 \dots 01\rangle$
2. After first set of Hadamards: $|+\rangle^{\otimes n} \otimes |-\rangle = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} |c\rangle \otimes |-\rangle$
3. After applying oracle U : $U\left(|+\rangle^{\otimes n} \otimes |-\rangle\right) = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$
4. After final set of Hadamards:

$$\begin{aligned} & (H^{\otimes n} \otimes I) \left(\frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \right) \\ &= \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} \left(\frac{1}{2^{n/2}} \sum_{m=0}^{2^n-1} (-1)^{c \cdot m} |m\rangle \right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \\ &= \frac{1}{2^n} \sum_{c=0}^{2^n-1} \sum_{m=0}^{2^n-1} (-1)^{f(c) + c \cdot m} |m\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \end{aligned}$$

Deutsch-Jozsa algorithm: Deutsch's algorithm for the $n > 1$ case

Output of circuit is 0 iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle^{\otimes n} \otimes |1\rangle = |0 \dots 0\rangle |1\rangle = |0 \dots 01\rangle$
2. After first set of Hadamards: $|+\rangle^{\otimes n} \otimes |-\rangle = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} |c\rangle \otimes |-\rangle$
3. After applying oracle U : $U\left(|+\rangle^{\otimes n} \otimes |-\rangle\right) = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$
4. After final set of Hadamards: $(H^{\otimes n} \otimes I) \left(\frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \right) = \frac{1}{2^n} \sum_{c=0}^{2^n-1} \sum_{m=0}^{2^n-1} (-1)^{f(c)+c \cdot m} |m\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$
5. Amplitude of upper register being $|m\rangle = |0\rangle$:

$$\frac{1}{2^n} \sum_{c=0}^{2^n-1} (-1)^{f(c)}$$

Deutsch-Jozsa algorithm: Deutsch's algorithm for the $n > 1$ case

Output of circuit is 0 iff f is constant

1. Initial state: $|c\rangle \otimes |t\rangle = |0\rangle^{\otimes n} \otimes |1\rangle = |0 \dots 0\rangle |1\rangle = |0 \dots 01\rangle$
2. After first set of Hadamards: $|+\rangle^{\otimes n} \otimes |-\rangle = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} |c\rangle \otimes |-\rangle$
3. After applying oracle U : $U\left(|+\rangle^{\otimes n} \otimes |-\rangle\right) = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$
4. After final set of Hadamards: $(H^{\otimes n} \otimes I) \left(\frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} (-1)^{f(c)} |c\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \right) = \frac{1}{2^n} \sum_{c=0}^{2^n-1} \sum_{m=0}^{2^n-1} (-1)^{f(c)+c \cdot m} |m\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$
5. Amplitude of upper register being $|m\rangle = |0\rangle$: $\frac{1}{2^n} \sum_{c=0}^{2^n-1} (-1)^{f(c)}$
6. Probability of measuring upper register to get $m = 0$:

$$\left| \frac{1}{2^n} \sum_{c=0}^{2^n-1} (-1)^{f(c)} \right|^2 = \begin{cases} |(-1)^{f(c)}|^2 = 1 & \text{if } f \text{ is constant} \\ 0 & \text{if } f \text{ is balanced} \end{cases}$$

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- Lemma: the Hadamard transform

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Exercises

Exercises for the scoreboard:

New meta after Midterm 1

Score calculation formula is now $\frac{\text{multiplier} * \text{upvotes}}{\sqrt{\text{team size}}}$

Multiplier 4 problems

Implement one of these protocols or algorithms in a Python notebook, using a library such as Qiskit or Cirq, shared using Google Collaboratory, following the assignment directions:

1. Quantum dense coding
2. Quantum teleportation
3. Deutsch's algorithm, defined on two qubits

$$|\overline{\Phi}^t\rangle \left\{ \begin{array}{l} ZZ \rightarrow 00, 11 \\ XX \rightarrow \pm\pm, -- \\ YY \rightarrow ii, -i-i \\ ZX \rightarrow \underline{0+ \quad 0- \quad 1+ \quad 1-} \end{array} \right.$$