

# A periodic table of quantum error correction codes

Yipeng Huang

Rutgers University

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# Quantum error correction codes: theory and practical systems

## Theory cares about

- ▶ Code taxonomy
- ▶ Code parameters
- ▶ Code specification

## Practical systems need

- ▶ Code words
- ▶ Encoding circuit, and its compilation
- ▶ Logical operators, and their compilation
- ▶ Error syndrome parity check circuit, and its compilation
- ▶ Decoding algorithm

# What this talk is about

- ▶ A tutorial on quantum Reed-Muller codes, a taxonomy where the existence of logical operators is the most clear
- ▶ A geometric approach to establishing what operators are available, and its impact on compilation
- ▶ Stabilizers and codewords of quantum Reed-Muller codes, and their potential as symmetry-preserving quantum ansatzes.
- ▶ A correct generalization of the geometric view of quantum Reed-Muller codes to qudits.

# A periodic table of classical Reed-Muller codes

				1												
				$RM(-1, 0)$ [1, 0, $\infty$ ]		$RM(0, 0)$ [1, 1, 1]										
				1		1										
				$RM(-1, 1)$ [2, 0, $\infty$ ]		$RM(0, 1)$ [2, 1, 2]		$RM(1, 1)$ [2, 2, 1]								
				1		2		1								
				$RM(-1, 2)$ [4, 0, $\infty$ ]		$RM(0, 2)$ [4, 1, 4]		$RM(1, 2)$ [4, 3, 2]		$RM(2, 2)$ [4, 4, 1]						
				1		3		3		1						
				$RM(-1, 3)$ [8, 0, $\infty$ ]		$RM(0, 3)$ [8, 1, 8]		$RM(1, 3)$ [8, 4, 4]		$RM(2, 3)$ [8, 7, 2]		$RM(3, 3)$ [8, 8, 1]				
				1		4		6		4		1				
				$RM(-1, 4)$ [16, 0, $\infty$ ]		$RM(0, 4)$ [16, 1, 16]		$RM(1, 4)$ [16, 5, 8]		$RM(2, 4)$ [16, 11, 4]		$RM(3, 4)$ [16, 15, 2]		$RM(4, 4)$ [16, 16, 1]		
				1		5		10		10		5		1		
				$RM(-1, 5)$ [32, 0, $\infty$ ]		$RM(0, 5)$ [32, 1, 32]		$RM(1, 5)$ [32, 6, 16]		$RM(2, 5)$ [32, 16, 8]		$RM(3, 5)$ [32, 26, 4]		$RM(4, 5)$ [32, 31, 2]		$RM(5, 5)$ [32, 32, 1]

Trivial codes

Repetition codes

Punctured Hadamard codes

Self-dual codes

Extended H

Single Parity

Universe cc

$$RM(r = 0, m = 0)$$

A single classical bit

$$y = Gx$$

- ▶ Messages  $x$
- ▶ Codewords  $y$
- ▶ Generator matrix  $G = [1]$

A  $[n = 1, k = 1, d = 1]$  code

- ▶  $n$
- ▶  $k$
- ▶  $d$

$$x: \{ \tau_0, \tau_1 \}$$

$$y: \{ \tau_0, \tau_1 \}$$

$$RM(r = 1, m = 1) \in [n = 2, k = 2, d = 1]$$

Two classical bits

$$2^m = n$$

$$y = Gx$$

▶ Messages  $x \in \{[0, 0], [0, 1], [1, 0], [1, 1]\}$

▶ Codewords  $y \in \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

▶ Generator matrix  $G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$m$  {  $d_0$

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

A  $[n = 2, k = 2, d = 1]$  code

- ▶  $n$
- ▶  $k$
- ▶  $d$

$$G = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$y = x \cdot G$$

$$x \cdot G = y$$

$$\begin{bmatrix} 0, 0 \end{bmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 0, 0 \end{bmatrix}$$

$$\begin{bmatrix} 0, 1 \end{bmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 0, 1 \end{bmatrix}$$

$$\begin{bmatrix} 1, 0 \end{bmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1, 1 \end{bmatrix}$$

$$\begin{bmatrix} 1, 1 \end{bmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1, 0 \end{bmatrix}$$

$$RM(r = 0, m = 1) \in [n = 2, k = 1, d = 2]$$

$$y = Gx$$

- ▶ Messages  $x \in \{[0], [1]\}$
- ▶ Codewords  $y \in \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$
- ▶ Generator matrix  $G = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , the message and parity check bits

$$Hy \equiv 0 \pmod{2}$$

- ▶ Indication that no error was detected
- ▶ Parity check matrix  $H = [1 \quad 1]$

The only single parity-check, repetition code

- ▶  $G = H^T$ , so it is also self-dual

$R_m(r=0, m=1)$

$f(x_0) = x_0^0$

$$G = \begin{bmatrix} 1 & 1 \\ k & 1 \end{bmatrix}$$

$$H^T = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$P = QG$$

$$Q = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ k & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ k & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$n_i = 2$$

$$n_i = 2^m$$

$$k_i = 1$$

$$d_i = 2$$

$$R_m(r=1, m=2) : [4, 3, 2]$$

$$K: 2^m = 2^2 = 4$$

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{matrix} 4 \\ \\ \\ \end{matrix} \left. \vphantom{\begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{matrix}} \right\} \begin{matrix} f(x_0, x_1) = 1 \\ f(x_0, x_1) = x_0 \\ f(x_0, x_1) = x_1 \end{matrix}$$

$$[000] G = [0000]$$

$$[001] G = [0101]$$

$$[010] G = [0011]$$

$$[011] G = [0110]$$

$$[100] G =$$

$$[101] G =$$

$$[110] G =$$

$$[111] G =$$

$$RM(r = 1, m = 3) \in [n = 8, k = 4, d = 4]$$

Usually, we start with this code

- ▶ Smallest non-repetition error correcting code
- ▶ Is an extended Hamming code
- ▶ Is a truncated Hadamard code
- ▶ Closely related  $[7, 4, 3]$  saturates the Hamming bound.

$$R_m(r=1, m=3) = [8, 4, 4]$$

$$n = 2^m = 2^3 = 8$$

$\delta$

$$G = k \left[ \begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right] \left. \begin{array}{l} f(x_0, x_1, x_2) \quad 3c0 \\ f() = x_0 \\ f() = x_1 \\ f() = x_2 \end{array} \right\} 3c1$$

$$G = H^T$$

$H_q$

Reed-Solomon

Turbo codes

LDPC

# Fundamental differences between ECC and QECC

## No-cloning theorem

There is no way to duplicate an arbitrary quantum state. Suppose a cloning operation  $U_c$  exists. Then:



$$U_c(|\phi\rangle \otimes |\omega\rangle) = |\phi\rangle \otimes |\phi\rangle,$$

$$U_c(|\psi\rangle \otimes |\omega\rangle) = |\psi\rangle \otimes |\psi\rangle,$$

for arbitrary states  $|\phi\rangle, |\psi\rangle$  we wish to copy.

- ▶ The overlap of the final states is:

$$\langle\phi| \otimes \langle\phi| |\psi\rangle \otimes |\psi\rangle = \langle\phi| |\psi\rangle \cdot \langle\phi| |\psi\rangle = (\langle\phi| |\psi\rangle)^2$$

- ▶ The overlap of the final states is also:

$$\langle\phi| \otimes \langle\phi| |\psi\rangle \otimes |\psi\rangle = \langle\phi| \otimes \langle\omega| U^\dagger U |\psi\rangle \otimes |\omega\rangle = \langle\phi| \otimes \langle\omega| |\psi\rangle \otimes |\omega\rangle = \langle\phi| |\psi\rangle$$

- ▶  $(\langle\phi| |\psi\rangle)^2 = \langle\phi| |\psi\rangle$ , so  $\langle\phi| |\psi\rangle = 0$ , or  $\langle\phi| |\psi\rangle = 1$ ,  $|\phi\rangle$  and  $|\psi\rangle$  cannot be arbitrary states as claimed.

# One-qubit quantum (stabilized) states

$RM(-1, 0)$ $[1, 0, \infty]$	1	$RM(0, 0)$ $[1, 1, 1]$
$QRM_0(-1, -1)$ $[[1, 0, 1]]$ $S = \{Z\}$ codeword = $ 0\rangle$ logicals = $\emptyset$		$QRM_0(0, 0)$ $[[1, 0, 1]]$ $S = \{X\}$ codeword = $ +\rangle$ logicals = $\emptyset$
	$QRM_0(-1, 0)$ $[[1, 1, 1]]$ $S = \{\}$ codeword = $ \psi\rangle$ logicals = $U$	

# Two-qubit quantum error "codes"

$RM(-1, 1)$ [2, 0, $\infty$ ]	1	$RM(0, 1)$ [2, 1, 2]	1	$RM(1, 1)$ [2, 2, 1]
$QRM_1(-1, -1)$ [[2, 0, 1]] $S = \{Z_0, Z_1\}$  codeword= $ 0\rangle  0\rangle$ logicals= $\emptyset$		$QRM_1(0, 0)$ [[2, 0, 2]] $S =$ $\{Z_0 Z_1, X_0 X_1\}$ codeword= $ \Phi^+\rangle$  logicals= $\emptyset$		$QRM_1(1, 1)$ [[2, 0, 1]] $S = \{X_0, X_1\}$  codeword= $ +\rangle  +\rangle$ logicals= $\emptyset$
	$QRM_1(-1, 0)$ [[2, 1, 1]] $S = \{Z_0 Z_1\}$ logicals= $\{XX, Z, S, T, \dots\}$		$QRM_1(0, 1)$ [[2, 1, 1]] $S = \{X_0 X_1\}$ logicals= $\{ZZ, X, \sqrt{X}, \sqrt[4]{X}, \dots\}$	
		$QRM_1(-1, 1)$ [[2, 2, 1]] $S = \{\}$ logicals= $\{H, Z, ZZ, SS, X, XX, \sqrt{X}\sqrt{X}\}$		

$$QRM_{m=1}(q = -1, r = 0) \in [[n = 2, k = 1, d = 1]]$$

## A first attempt at making a QECC

- ▶ The code's stabilizer generators
- ▶ A valid choice of codewords
- ▶ Encoding circuit
- ▶ Parity check error syndrome extraction circuit

$$QRM_{m=1}(q = 0, r = 1) \in [[n = 2, k = 1, d = 1]]$$

## A second attempt at making a QECC

- ▶ The code's stabilizer generators
- ▶ A valid choice of codewords
- ▶ Encoding circuit
- ▶ Parity check error syndrome extraction circuit

## Degrees of freedom irrelevant to theory but impactful in systems

- ▶ Choice of codewords
- ▶ Encoding circuit
- ▶ Parity check circuit

# Four-qubit quantum error (detection) codes

$RM(-1, 2)$ [4, 0, $\infty$ ]	1	$RM(0, 2)$ [4, 1, 4]	2	$RM(1, 2)$ [4, 3, 2]	1	$RM(2, 2)$ [4, 4, 1]
$QRM_2(-1, -1)$ [[4, 0, 1]] $S =$ $\{Z_0, Z_1, Z_2, Z_3\}$  codeword= $ 0\rangle  0\rangle  0\rangle  0\rangle$ logicals= $\emptyset$		$QRM_2(0, 0)$ [[4, 0, 2]] $S =$ $\{Z_0 Z_1, Z_1 Z_2,$ $Z_2 Z_3, Z_3 Z_0$ $X_0 X_1 X_2 X_3\}$  logicals= $\emptyset$		$QRM_2(1, 1)$ [[4, 0, 2]] $S =$ $\{Z_0 Z_1 Z_2 Z_3,$ $X_0 X_1, X_1 X_2,$ $X_2 X_3, X_3 X_0\}$  logicals= $\emptyset$		$QRM_2(2, 2)$ [[4, 0, 1]] $S =$ $\{X_0, X_1, X_2, X_3\}$  codeword= $ +\rangle  +\rangle  +\rangle  +\rangle$ logicals= $\emptyset$
	$QRM_2(-1, 0)$ [[4, 1, 1]] $S =$ $\{Z_0 Z_1, Z_1 Z_2,$ $Z_2 Z_3, Z_3 Z_0\}$ logicals= $\{XXXX, Z, S, T, \dots\}$		$QRM_2(0, 1)$ [[4, 2, 2]] $S =$ $\{Z_0 Z_1 Z_2 Z_3,$ $X_0 X_1 X_2 X_3\}$ logicals= $\{H, ZZ, SSSS, XX, \sqrt{X}\sqrt{X}\sqrt{X}\sqrt{X}\}$		$QRM_2(1, 2)$ [[4, 1, 1]] $S =$ $\{X_0 X_1, X_1 X_2,$ $X_2 X_3, X_3 X_0\}$ logicals= $\{ZZZZ, X, \sqrt{X}, \sqrt[4]{X}, \dots\}$	
		$QRM_2(-1, 1)$ [[4, 3, 1]] $S =$ $\{Z_0 Z_1 Z_2 Z_3\}$ logicals= $\{Z, ZZ, SS, SSSS, TTTT, XX, XXXX\}$		$QRM_2(0, 2)$ [[4, 3, 1]] $S =$ $\{X_0 X_1 X_2 X_3\}$ logicals= $\{ZZ, ZZZZ, X, XX, \sqrt{X}\sqrt{X}, \sqrt{X}\sqrt{X}\sqrt{X}\sqrt{X}, \sqrt[4]{X}\sqrt[4]{X}\sqrt[4]{X}\sqrt[4]{X}\}$		
			$QRM_2(-1, 2)$ [[4, 4, 1]] $S = \{\}$ logicals= $\{H, Z, ZZ, ZZZZ, SSSS, X, XX, XXXX, \sqrt{X}\sqrt{X}\sqrt{X}\sqrt{X}\}$			

$$QRM_{m=2}(q = 0, r = 1) \in [[n = 4, k = 2, d = 2]]$$

The smallest quantum error detecting code

Codes-as-ansatz

# Eight-qubit quantum error (correction) codes

$RM(-1, 3)$ [8, 0, $\infty$ ]	1	$RM(0, 3)$ [8, 1, 8]	3	$RM(1, 3)$ [8, 4, 4]	3	$RM(2, 3)$ [8, 7, 2]	1	$RM(3, 3)$ [8, 8, 1]
$QRM_3(-1, -1)$ [[8, 0, 1]] $S : 8Z^{\otimes 1}$  codeword= $ 0\rangle^{\otimes 8}$ logicals= $\emptyset$		$QRM_3(0, 0)$ [[8, 0, 2]] $S : 12Z^{\otimes 2}, 1X^{\otimes 8}$  logicals= $\emptyset$		$QRM_3(1, 1)$ [[8, 0, 4]] $S : 6Z^{\otimes 4}, 6X^{\otimes 4}$  logicals= $\emptyset$		$QRM_3(2, 2)$ [[8, 0, 2]] $S : 1Z^{\otimes 8}, 12X^{\otimes 2}$  logicals= $\emptyset$		$QRM_3(3, 3)$ [[8, 0, 1]] $S : 8X^{\otimes 1}$  codeword= $ +\rangle^{\otimes 8}$ logicals= $\emptyset$
	$QRM_3(-1, 0)$ [[8, 1, 1]] $S : 12Z^{\otimes 2}$  logicals= $\{Z, S, T, \dots\}$		$QRM_3(0, 1)$ [[8, 3, 2]] $S : 6Z^{\otimes 4}, 1X^{\otimes 8}$ logicals= $\{Z^{\otimes 2}, S^{\otimes 4}, T^{\otimes 8}\}$		$QRM_3(1, 2)$ [[8, 3, 2]] $S : 1Z^{\otimes 8}, 6X^{\otimes 4}$ logicals= $\{Z^{\otimes 4}\}$		$QRM_3(2, 3)$ [[8, 1, 1]] $S : 12X^{\otimes 2}$  logicals= $\{Z^{\otimes 8}\}$	
		$QRM_3(-1, 1)$ [[8, 4, 1]] $S : 6Z^{\otimes 4}$  logicals= $\{Z^{\otimes 1,2}, S^{\otimes 2,4}, T^{\otimes 4,8}\}$		$QRM_3(0, 2)$ [[8, 6, 2]] $S : 1Z^{\otimes 8}, 1X^{\otimes 8}$ logicals= $\{H, Z^{\otimes 2}, Z^{\otimes 4}, S^{\otimes 8}\}$		$QRM_3(1, 3)$ [[8, 4, 1]] $S : 6X^{\otimes 4}$  logicals= $\{Z^{\otimes 4}, Z^{\otimes 8}\}$		
			$QRM_3(-1, 2)$ [[8, 7, 1]] $S : 1 \times 8Z$ logicals= $\{Z, Z^{\otimes 2}, Z^{\otimes 4}, S^{\otimes 4}, S^{\otimes 8}\}$		$QRM_3(0, 3)$ [[8, 7, 1]] $S : 1 \times 8X$ logicals= $\{Z^{\otimes 2}, Z^{\otimes 4}, Z^{\otimes 8}\}$			
				$QRM_3(-1, 3)$ [[8, 8, 1]] $S = \{\}$ logicals= $\{H, Z, Z^{\otimes 2}, Z^{\otimes 4}, Z^{\otimes 8}, S^{\otimes 8}\}$				

# 16-qubit quantum error correction codes

$RM(-1, 4)$ [16, 0, $\infty$ ]	1	$RM(0, 4)$ [16, 1, 16]	4	$RM(1, 4)$ [16, 5, 8]	6	$RM(2, 4)$ [16, 11, 4]	4	$RM(3, 4)$ [16, 15, 2]	1	$RM(4, 4)$ [16, 16, 1]
$QRM_4(-1, -1)$ [[16, 0, 1]] S : $16Z^{\otimes 1}$ logicals= $\emptyset$		$QRM_4(0, 0)$ [[16, 0, 2]] S : $32Z^{\otimes 2}, 1X^{\otimes 16}$ logicals= $\emptyset$		$QRM_4(1, 1)$ [[16, 0, 4]] S : $24Z^{\otimes 4}, 8X^{\otimes 8}$ logicals= $\emptyset$		$QRM_4(2, 2)$ [[16, 0, 4]] S : $8Z^{\otimes 8}, 24X^{\otimes 4}$ logicals= $\emptyset$		$QRM_4(3, 3)$ [[16, 0, 2]] S : $1Z^{\otimes 16}, 32X^{\otimes 2}$ logicals= $\emptyset$		$QRM_4(4, 4)$ [[16, 0, 1]] S : $16X^{\otimes 1}$ logicals= $\emptyset$
	$QRM_4(-1, 0)$ [[16, 1, 1]] S : $32Z^{\otimes 2}$ logicals= {Z, S, T, ...}	$QRM_4(0, 1)$ [[16, 4, 2]] S : $24Z^{\otimes 4}, 1X^{\otimes 16}$ logicals= {Z <sup>⊗2</sup> , S <sup>⊗4</sup> , T <sup>⊗8</sup> , $\sqrt{T}^{\otimes 16}$ }		$QRM_4(1, 2)$ [[16, 6, 4]] S : $8Z^{\otimes 8}, 8X^{\otimes 8}$ logicals= {H, Z <sup>⊗4</sup> , S <sup>⊗16</sup> }		$QRM_4(2, 3)$ [[16, 4, 2]] S : $1Z^{\otimes 16}, 24X^{\otimes 4}$ logicals= {Z <sup>⊗8</sup> }		$QRM_4(3, 4)$ [[16, 1, 1]] S : $32X^{\otimes 2}$ logicals= {Z <sup>⊗16</sup> }		
		$QRM_4(-1, 1)$ [[16, 5, 1]] logicals= {Z <sup>⊗1,2</sup> , S <sup>⊗2,4</sup> , T <sup>⊗4,8</sup> , $\sqrt{T}^{\otimes 8,16}$ , $\sqrt[4]{T}^{\otimes 16}$ }		$QRM_4(0, 2)$ [[16, 10, 2]] logicals= {Z <sup>⊗2</sup> , Z <sup>⊗4</sup> , S <sup>⊗8</sup> , S <sup>⊗16</sup> }		$QRM_4(1, 3)$ [[16, 10, 2]] logicals= {Z <sup>⊗4</sup> , Z <sup>⊗8</sup> }		$QRM_4(2, 4)$ [[16, 5, 1]] logicals= {Z <sup>⊗8</sup> , Z <sup>⊗16</sup> }		
		$QRM_4(-1, 2)$ [[16, 11, 1]] logicals= {Z <sup>⊗1,2,4</sup> , S <sup>⊗4,8,16</sup> , T <sup>⊗16</sup> }		$QRM_4(0, 3)$ [[16, 14, 2]] logicals= {H, Z <sup>⊗2,4,8</sup> , S <sup>⊗16</sup> }		$QRM_4(1, 4)$ [[16, 11, 1]] logicals= {Z <sup>⊗4</sup> , Z <sup>⊗8</sup> , Z <sup>⊗16</sup> }				
			$QRM_4(-1, 3)$ [[16, 15, 1]] logicals= {Z <sup>⊗1,2,4,8</sup> , S <sup>⊗8,16</sup> }		$QRM_4(0, 4)$ [[16, 15, 1]] logicals= {Z <sup>⊗2</sup> , Z <sup>⊗4</sup> , Z <sup>⊗8</sup> , Z <sup>⊗16</sup> }					
				$QRM_4(-1, 4)$ [[16, 16, 1]] logicals= {H, Z, Z <sup>⊗2</sup> , Z <sup>⊗4</sup> , Z <sup>⊗8</sup> , Z <sup>⊗16</sup> , S <sup>⊗16</sup> }						

$$QRM_{m=3}(q = 1, r = 1) \in [[n = 8, k = 0, d = 4]]$$

Our first encounter with a quantum error correcting code

The geometric construction of quantum Reed-Muller codes

The duality of non-Clifford operations

What about non-Reed-Muller codes?

- ▶ Modified Reed-Muller codes.  $[[7, 1, 3]]$ , a code with transversal Cliffords.
- ▶ Non-CSS codes.  $[[5, 1, 3]]$ , the smallest qubit QECC

# A periodic table of classical trit generalized Reed-Muller codes

			$GRM_3(-1, 0)$ $[1, 0, \infty]_3$	$GRM_3(0, 0)$ $[1, 1, 1]_3$			
		$GRM_3(-1, 1)$ $[3, 0, \infty]_3$	$GRM_3(0, 1)$ $[3, 1, 3]_3$	$GRM_3(1, 1)$ $[3, 2, 2]_3$	$GRM_3(2, 1)$ $[3, 3, 1]_3$		
	$GRM_3(-1, 2)$ $[9, 0, \infty]_3$	$GRM_3(0, 2)$ $[9, 1, 9]_3$	$GRM_3(1, 2)$ $[9, 3, 6]_3$	$GRM_3(2, 2)$ $[9, 6, 3]_3$	$GRM_3(3, 2)$ $[9, 8, 2]_3$	$GRM_3(4, 2)$ $[9, 9, 1]_3$	
$GRM_3(-1, 3)$ $[27, 0, \infty]_3$	$GRM_3(0, 3)$ $[27, 1, 27]_3$	$GRM_3(1, 3)$ $[27, 4, 18]_3$	$GRM_3(2, 3)$ $[27, 10, 9]_3$	$GRM_3(3, 3)$ $[27, 17, 6]_3$	$GRM_3(4, 3)$ $[27, 23, 3]_3$	$GRM_3(5, 3)$ $[27, 26, 2]_3$	$GRM_3(6, 3)$ $[27, 27, 1]_3$

Trivial codes

Repetition codes

Single parity-check codes

Universe codes

# One-qutrit quantum (stabilized) states

$GRM(-1, 0)$ $[1, 0, \infty]_3$	1	$GRM(0, 0)$ $[1, 1, 1]_3$
$QGRM_0^3(-1, -1)$ $[[1, 0, 1]]_3$ $S = \left\{ Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} \right\}$ codeword = $ 0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ logicals = $\emptyset$		$QGRM_0^3(0, 0)$ $[[1, 0, 1]]_3$ $S = \left\{ X = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}$ codeword = $ +\rangle = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ logicals = $\emptyset$
	$QGRM_0^3(-1, 0)$ $[[1, 1, 1]]_3$ $S = \{ \}$ codeword = $ \psi\rangle$ logicals = $U$	

# Three-qutrit quantum error (detection) codes

$GRM_3(-1, 1)$ [[3, 0, ∞]] <sub>3</sub>	1	$GRM_3(0, 1)$ [[3, 1, 3]] <sub>3</sub>	1	$GRM_3(1, 1)$ [[3, 2, 2]] <sub>3</sub>	1	$GRM_3(2, 1)$ [[3, 3, 1]] <sub>3</sub>
$QGRM_1^3(-1, -1)$ [[3, 0, 1]] <sub>3</sub> $S = \{Z_0, Z_1, Z_2\}$  codeword= $ 0\rangle  0\rangle  0\rangle$ logicals= $\emptyset$		$QGRM_1^3(0, 0)$ [[3, 0, 2]] <sub>3</sub> $S = \{Z_0 Z_1, Z_1 Z_2, Z_2 Z_0, X_0 X_1 X_2\}$  logicals= $\emptyset$		$QGRM_1^3(1, 1)$ [[3, 0, 2]] <sub>3</sub> $S = \{Z_0 Z_1 Z_2, X_0 X_1, X_1 X_2, X_2 X_0\}$  logicals= $\emptyset$		$QGRM_1^3(2, 2)$ [[3, 0, 1]] <sub>3</sub> $S = \{X_0, X_1, X_2\}$  codeword= $ +\rangle  +\rangle  +\rangle$ logicals= $\emptyset$
	$QGRM_1^3(-1, 0)$ [[3, 1, 1]] <sub>3</sub> $S = \{Z_0 Z_1, Z_1 Z_2, Z_2 Z_0\}$ logicals= $\{XXX, Z, S, T, \dots\}$		$QGRM_1^3(0, 1)$ [[3, 1, 2]] <sub>3</sub> $S = \{Z_0 Z_1 Z_2, X_0 X_1 X_2\}$ logicals= $\{H, ZZ, SSS, XX, \sqrt{X}\sqrt{X}\sqrt{X}\}$		$QGRM_1^3(1, 2)$ [[3, 1, 1]] <sub>3</sub> $S = \{X_0 X_1, X_1 X_2, X_2 X_0\}$ logicals= $\{ZZZ, X, \sqrt{X}, \sqrt[4]{X}, \dots\}$	
		$QGRM_1^3(-1, 1)$ [[3, 2, 1]] <sub>3</sub> $S = \{Z_0 Z_1 Z_2\}$ logicals= $\{Z, ZZ, SS, SSS, XX, XXX\}$		$QGRM_1^3(0, 2)$ [[3, 2, 1]] <sub>3</sub> $S = \{X_0 X_1 X_2\}$ logicals= $\{ZZ, ZZZ, X, XX, \sqrt{X}\sqrt{X}, \sqrt{X}\sqrt{X}\sqrt{X}\}$		
			$QGRM_1^3(-1, 2)$ [[3, 3, 1]] <sub>3</sub> $S = \{\}$ logicals= $\{H, Z, ZZ, ZZZ, SSS, X, XX, XXX, \sqrt{X}\sqrt{X}\sqrt{X}\}$			

# Nine-qutrit quantum error (correction) codes

$GRM_3(-1, 2)$ [9, 0, $\infty$ ] <sub>3</sub>	1	$GRM_3(0, 2)$ [9, 1, 9] <sub>3</sub>	2	$GRM_3(1, 2)$ [9, 3, 6] <sub>3</sub>	3	$GRM_3(2, 2)$ [9, 6, 3] <sub>3</sub>	2	$GRM_3(3, 2)$ [9, 8, 2] <sub>3</sub>	1	$GRM_3(4, 2)$ [9, 9, 1] <sub>3</sub>
$QGRM_2^3(-1, -1)$ [[9, 0, 1]] <sub>3</sub> $S =$ $9Z^{\otimes 1}$ logicals= $\emptyset$		$QGRM_2^3(0, 0)$ [[9, 0, 2]] <sub>3</sub> $S =$ $?Z^{\otimes 2}, 1X^{\otimes 9}$ logicals= $\emptyset$		$QGRM_2^3(1, 1)$ [[9, 0, 3]] <sub>3</sub> $S =$ $6Z^{\otimes 3}, ?X^{\otimes 6}$ logicals= $\emptyset$		$QGRM_2^3(2, 2)$ [[9, 0, 3]] <sub>3</sub> $S =$ $?Z^{\otimes 6}, 6X^{\otimes 3}$ logicals= $\emptyset$		$QGRM_2^3(3, 3)$ [[9, 0, 2]] <sub>3</sub> $S =$ $1Z^{\otimes 9}, ?X^{\otimes 2}$ logicals= $\emptyset$		$QGRM_2^3(4, 4)$ [[9, 0, 1]] <sub>3</sub> $S =$ $9X^{\otimes 1}$ logicals= $\emptyset$
$QGRM_2^3(-1, 0)$ [[9, 1, 1]] <sub>3</sub> $S =$ $?Z^{\otimes 2}$ logicals= {Z, S, T, ...}		$QGRM_2^3(0, 1)$ [[9, 2, 2]] <sub>3</sub> $S =$ $6Z^{\otimes 3}, 1X^{\otimes 9}$ logicals= {Z <sup>⊗2</sup> , S <sup>⊗3</sup> , T <sup>⊗9</sup> }		$QGRM_2^3(1, 2)$ [[9, 3, 3]] <sub>3</sub> $S =$ $?Z^{\otimes 6}, ?X^{\otimes 6}$ logicals= {H, Z <sup>⊗3</sup> , S <sup>⊗9</sup> }		$QGRM_2^3(2, 3)$ [[9, 2, 2]] <sub>3</sub> $S =$ $1Z^{\otimes 9}, 6X^{\otimes 3}$ logicals= {Z <sup>⊗6</sup> }		$QGRM_2^3(3, 4)$ [[9, 1, 1]] <sub>3</sub> $S =$ $?X^{\otimes 2}$ logicals= {Z <sup>⊗9</sup> }		
		$QGRM_2^3(-1, 1)$ [[9, 3, 1]] <sub>3</sub> logicals= {Z <sup>⊗1,2</sup> , S <sup>⊗2,3</sup> , T <sup>⊗6,9</sup> }		$QGRM_2^3(0, 2)$ [[9, 5, 2]] <sub>3</sub> logicals= {Z <sup>⊗2,3</sup> , S <sup>⊗6,9</sup> }		$QGRM_2^3(1, 3)$ [[9, 5, 2]] <sub>3</sub> logicals= {Z <sup>⊗3,6</sup> }		$QGRM_2^3(2, 4)$ [[9, 3, 1]] <sub>3</sub> logicals= {Z <sup>⊗6,9</sup> }		
		$QGRM_2^3(-1, 2)$ [[9, 6, 1]] <sub>3</sub> logicals= {Z <sup>⊗1,2,3</sup> , S <sup>⊗3,6,9</sup> }		$QGRM_2^3(0, 3)$ [[9, 7, 2]] <sub>3</sub> logicals= {H, Z <sup>⊗2,3,6</sup> , S <sup>⊗9</sup> }		$QGRM_2^3(1, 4)$ [[9, 6, 1]] <sub>3</sub> logicals= {Z <sup>⊗3,6,9</sup> }				
				$QGRM_2^3(-1, 3)$ [[9, 8, 1]] <sub>3</sub> logicals= {Z <sup>⊗1,2,3,6</sup> , S <sup>⊗6,9</sup> }		$QGRM_2^3(0, 4)$ [[9, 8, 1]] <sub>3</sub> logicals= {Z <sup>⊗2,3,6,9</sup> }				
				$QGRM_2^3(-1, 4)$ [[9, 9, 1]] <sub>3</sub> logicals= {H, Z <sup>⊗1,2,3,6,9</sup> , S <sup>⊗9</sup> }						

# A periodic table of classical quart generalized Reed-Muller codes

				$GRM_4(-1, 0)$ $[1, 0, \infty]_4$	$GRM_4(0, 0)$ $[1, 1, 1]_4$			
		$GRM_4(-1, 1)$ $[4, 0, \infty]_4$	$GRM_4(0, 1)$ $[4, 1, 4]_4$	$GRM_4(1, 1)$ $[4, 2, 3]_4$	$GRM_4(2, 1)$ $[4, 3, 2]_4$	$GRM_4(3, 1)$ $[4, 4, 1]_4$		
$GRM_4(-1, 2)$ $[16, 0, \infty]_4$	$GRM_4(0, 2)$ $[16, 1, 16]_4$	$GRM_4(1, 2)$ $[16, 3, 12]_4$	$GRM_4(2, 2)$ $[16, 6, 8]_4$	$GRM_4(3, 2)$ $[16, 10, 4]_4$	$GRM_4(4, 2)$ $[16, 13, 3]_4$	$GRM_4(5, 2)$ $[16, 15, 2]_4$	$GRM_4(6, 2)$ $[16, 16, 1]_4$	

Trivial codes

Repetition codes

Self-dual code

Single parity-check codes

Universe codes