

A periodic table of quantum error correction codes

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Quantum error correction codes: theory and practical systems

Theory cares about

- ▶ Code taxonomy
- ▶ Code parameters
- ▶ Code specification

Practical systems need

- ▶ Code words
- ▶ Encoding circuit, and its compilation
- ▶ Logical operators, and their compilation
- ▶ Error syndrome parity check circuit, and its compilation
- ▶ Decoding algorithm

What this talk is about

- ▶ A tutorial on quantum Reed-Muller codes, a taxonomy where the existence of logical operators is the most clear
- ▶ A geometric approach to establishing what operators are available, and its impact on compilation
- ▶ Stabilizers and codewords of quantum Reed-Muller codes, and their potential as symmetry-preserving quantum ansatzes.
- ▶ A correct generalization of the geometric view of quantum Reed-Muller codes to qudits.

A periodic table of classical Reed-Muller codes

				1									
				$RM(-1, 0)$ [1, 0, ∞]		$RM(0, 0)$ [1, 1, 1]							
				1		1							
				$RM(-1, 1)$ [2, 0, ∞]		$RM(0, 1)$ [2, 1, 2]		$RM(1, 1)$ [2, 2, 1]					
				1		2		1					
				$RM(-1, 2)$ [4, 0, ∞]		$RM(0, 2)$ [4, 1, 4]		$RM(1, 2)$ [4, 3, 2]		$RM(2, 2)$ [4, 4, 1]			
				1		3		3		1			
				$RM(-1, 3)$ [8, 0, ∞]		$RM(0, 3)$ [8, 1, 8]		$RM(1, 3)$ [8, 4, 4]		$RM(2, 3)$ [8, 7, 2]	$RM(3, 3)$ [8, 8, 1]		
				1		4		6		4	1		
				$RM(-1, 4)$ [16, 0, ∞]		$RM(0, 4)$ [16, 1, 16]		$RM(1, 4)$ [16, 5, 8]		$RM(2, 4)$ [16, 11, 4]	$RM(3, 4)$ [16, 15, 2]	$RM(4, 4)$ [16, 16, 1]	
				1		5		10		10	5	1	
				$RM(-1, 5)$ [32, 0, ∞]		$RM(0, 5)$ [32, 1, 32]		$RM(1, 5)$ [32, 6, 16]		$RM(2, 5)$ [32, 16, 8]	$RM(3, 5)$ [32, 26, 4]	$RM(4, 5)$ [32, 31, 2]	$RM(5, 5)$ [32, 32, 1]

Trivial codes

Repetition codes

Punctured Hadamard codes

Self-dual codes

Extended H

Single Parity

Universe cc

$$RM(r = 0, m = 0)$$

A single classical bit

$$y = Gx$$

- ▶ Messages x
- ▶ Codewords y
- ▶ Generator matrix $G = [1]$

A $[n = 1, k = 1, d = 1]$ code

- ▶ n
- ▶ k
- ▶ d

$$x: \{ \tau_0, \tau_1 \}$$

$$y: \{ \tau_0, \tau_1 \}$$

$$RM(r = 1, m = 1) \in [n = 2, k = 2, d = 1]$$

Two classical bits

$$2^m = n$$

$$y = Gx$$

▶ Messages $x \in \{[0, 0], [0, 1], [1, 0], [1, 1]\}$

▶ Codewords $y \in \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

▶ Generator matrix $G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

m { d_0

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

A $[n = 2, k = 2, d = 1]$ code

- ▶ n
- ▶ k
- ▶ d

$$G = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$y = x \cdot G$$

$$x \cdot G = y$$

$$\begin{bmatrix} 0, 0 \end{bmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 0, 0 \end{bmatrix}$$

$$\begin{bmatrix} 0, 1 \end{bmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 0, 1 \end{bmatrix}$$

$$\begin{bmatrix} 1, 0 \end{bmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1, 1 \end{bmatrix}$$

$$\begin{bmatrix} 1, 1 \end{bmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1, 0 \end{bmatrix}$$

$$RM(r = 0, m = 1) \in [n = 2, k = 1, d = 2]$$

$$y = Gx$$

- ▶ Messages $x \in \{[0], [1]\}$
- ▶ Codewords $y \in \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$
- ▶ Generator matrix $G = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the message and parity check bits

$$Hy \equiv 0 \pmod{2}$$

- ▶ Indication that no error was detected
- ▶ Parity check matrix $H = [1 \ 1]$

The only single parity-check, repetition code

- ▶ $G = H^T$, so it is also self-dual

$R_m(r=0, m=1)$

$f(x_0) = x_0^0$

$$G = \begin{bmatrix} 1 & 1 \\ k & 1 \end{bmatrix}$$

$$H^T = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$P = QG$$

$$Q = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ k & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ k & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$n_i = 2$$

$$n_i = 2^m$$

$$k_i = 1$$

$$d_i = 2$$

$$R_m(r=1, m=2) : [4, 3, 2]$$

$$K: 2^m = 2^2 = 4$$

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{matrix} 4 \\ \\ \\ \end{matrix} \left. \begin{matrix} f(x_0, x_1) = 1 \\ f(x_0, x_1) = x_0 \\ f(x_0, x_1) = x_1 \end{matrix} \right\}$$

$$[000] G = [0000]$$

$$[001] G = [0101]$$

$$[010] G = [0011]$$

$$[011] G = [0110]$$

$$[100] G =$$

$$[101] G =$$

$$[110] G =$$

$$[111] G =$$

$$RM(r = 1, m = 3) \in [n = 8, k = 4, d = 4]$$

Usually, we start with this code

- ▶ Smallest non-repetition error correcting code
- ▶ Is an extended Hamming code
- ▶ Is a truncated Hadamard code
- ▶ Closely related $[7, 4, 3]$ saturates the Hamming bound.

$$R_m(r=1, m=3) = [8, 4, 4]$$

$$n = 2^m = 2^3 = 8$$

δ

$$G = k \left[\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right] \left. \begin{array}{l} f(x_0, x_1, x_2) \quad 3 \text{ c0} \\ f() = x_0 \\ f() = x_1 \\ f() = x_2 \end{array} \right\} 3 \text{ c1}$$

$$G = H^T$$

H_q

Reed-Solomon

Turbo codes

LDPC

3-qubit repetition code

9-qubit Shor QEC

5-qubit Perfect code

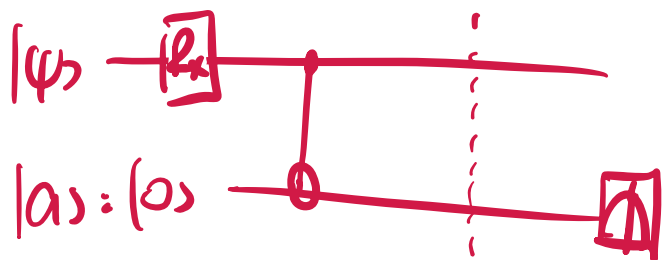
An error model

$$R_x(\theta) = \cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right) X$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$R_x(\theta)|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|\psi\rangle - i \sin\left(\frac{\theta}{2}\right) X|\psi\rangle$$

$$\cos\left(\frac{\theta}{2}\right) [|\psi\rangle \otimes |0\rangle] - i \sin\left(\frac{\theta}{2}\right) [X|\psi\rangle \otimes |1\rangle]$$



$$|\psi\rangle \otimes |0\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle$$

$$\xrightarrow{\text{CNOT}} \alpha|00\rangle + \beta|11\rangle$$

Stabilized States

Logical states

Codes

→ encoding

→ logical operations

→ syndrome extraction

Fundamental differences between ECC and QECC

No-cloning theorem

There is no way to duplicate an arbitrary quantum state. Suppose a cloning operation U_c exists. Then:



$$U_c(|\phi\rangle \otimes |\omega\rangle) = |\phi\rangle \otimes |\phi\rangle,$$

$$U_c(|\psi\rangle \otimes |\omega\rangle) = |\psi\rangle \otimes |\psi\rangle,$$

for arbitrary states $|\phi\rangle, |\psi\rangle$ we wish to copy.

- ▶ The overlap of the final states is:

$$\langle\phi| \otimes \langle\phi| |\psi\rangle \otimes |\psi\rangle = \langle\phi| |\psi\rangle \cdot \langle\phi| |\psi\rangle = (\langle\phi| |\psi\rangle)^2$$

- ▶ The overlap of the final states is also:

$$\langle\phi| \otimes \langle\phi| |\psi\rangle \otimes |\psi\rangle = \langle\phi| \otimes \langle\omega| U^\dagger U |\psi\rangle \otimes |\omega\rangle = \langle\phi| \otimes \langle\omega| |\psi\rangle \otimes |\omega\rangle = \langle\phi| |\psi\rangle$$

- ▶ $(\langle\phi| |\psi\rangle)^2 = \langle\phi| |\psi\rangle$, so $\langle\phi| |\psi\rangle = 0$, or $\langle\phi| |\psi\rangle = 1$, $|\phi\rangle$ and $|\psi\rangle$ cannot be arbitrary states as claimed.

One-qubit quantum (stabilized) states

$RM(-1, 0)$ $[1, 0, \infty]$	1	$RM(0, 0)$ $[1, 1, 1]$
$QRM_0(-1, -1)$ $[[1, 0, 1]]$ $S = \{Z\}$ codeword = $ 0\rangle$ logicals = \emptyset		$QRM_0(0, 0)$ $[[1, 0, 1]]$ $S = \{X\}$ codeword = $ +\rangle$ logicals = \emptyset
	$QRM_0(-1, 0)$ $[[1, 1, 1]]$ $S = \{\}$ codeword = $ \psi\rangle$ logicals = U	

Two-qubit quantum error "codes"

[1 1]

[2 2]

$RM(-1, 1)$ [2, 0, ∞]	1	$RM(0, 1)$ [2, 1, 2]	1	$RM(1, 1)$ [2, 2, 1]
$QRM_1(-1, -1)$ [[2, 0, 1]] $S = \{Z_0, Z_1\}$ zi z1 codeword= $ 0\rangle 0\rangle$ logicals= \emptyset		$QRM_1(0, 0)$ [[2, 0, 2]] $S =$ $\{Z_0 Z_1, X_0 X_1\}$ codeword= $ \Phi^+\rangle$ logicals= \emptyset		$QRM_1(1, 1)$ [[2, 0, 1]] $S = \{X_0, X_1\}$ codeword= $ +\rangle +\rangle$ logicals= \emptyset
	$QRM_1(-1, 0)$ [[2, 1, 1]] $S = \{Z_0 Z_1\}$ logicals= $\{XX, Z, S, T, \dots\}$		$QRM_1(0, 1)$ [[2, 1, 1]] $S = \{X_0 X_1\}$ logicals= $\{ZZ, X, \sqrt{X}, \sqrt[4]{X}, \dots\}$	
		$QRM_1(-1, 1)$ [[2, 2, 1]] $S = \{ \}$ logicals= $\{H, Z, ZZ, SS, X, XX, \sqrt{X}\sqrt{X}\}$		

$$QRM_{m=1}(q = -1, r = 0) \in [[n = 2, k = 1, d = 1]]$$

A first attempt at making a QECC

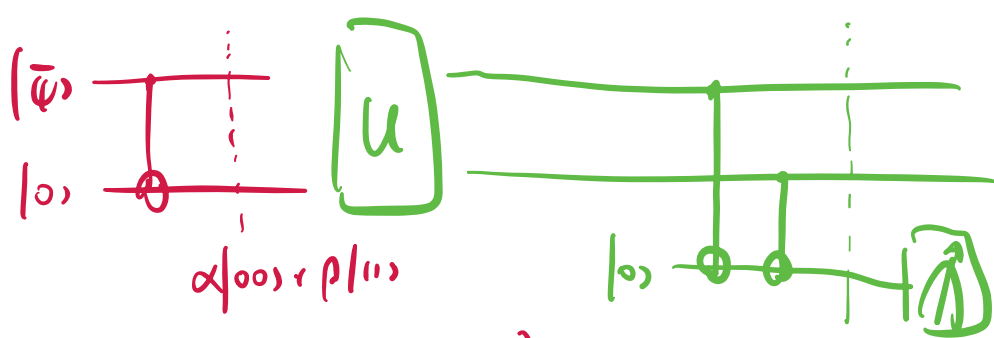
- ▶ The code's stabilizer generators
- ▶ A valid choice of codewords
- ▶ Encoding circuit
- ▶ Parity check error syndrome extraction circuit

$$ZZ^z \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \left\{ \begin{array}{l} ZZ^z |00\rangle \\ ZZ^z |11\rangle \end{array} \right.$$

$$|\bar{\psi}\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$= \alpha|00\rangle + \beta|11\rangle$$

▷ Logical operations



$$\begin{aligned} \bar{X}|\bar{\psi}\rangle &= \bar{X}(\alpha|00\rangle + \beta|11\rangle) \\ &= X X (\alpha|00\rangle + \beta|11\rangle) \\ &= \beta|00\rangle + \alpha|11\rangle \\ &= \bar{X}|\bar{\psi}\rangle \end{aligned}$$

$$\begin{aligned} &\alpha|00\rangle \otimes |0\rangle + \beta|11\rangle \otimes |0\rangle \\ &+ \gamma|01\rangle \otimes |1\rangle + \delta|10\rangle \otimes |1\rangle \end{aligned}$$

$$|\psi\rangle \longrightarrow |\psi'\rangle$$

	x_0	x_1	z_0	z_1
\bar{x}_0	1	1		
\bar{x}_1		1		
\bar{z}_0			1	
\bar{z}_1			1	1

$$\begin{aligned} \bar{Z}|\bar{\psi}\rangle &= \bar{Z}(\alpha|00\rangle + \beta|11\rangle) \\ &= Z I (\alpha|00\rangle + \beta|11\rangle) \\ &= \alpha|00\rangle - \beta|11\rangle \\ &= \bar{Z}|\bar{\psi}\rangle \end{aligned}$$

$$QRM_{m=1}(q = 0, r = 1) \in [[n = 2, k = 1, d = 1]]$$

A second attempt at making a QECC

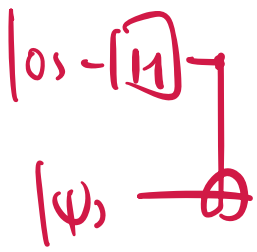
- ▶ The code's stabilizer generators
- ▶ A valid choice of codewords
- ▶ Encoding circuit
- ▶ Parity check error syndrome extraction circuit

$$XX = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\bar{I}_0 = |00\rangle + |11\rangle \quad \bar{I}_1 = |01\rangle + |10\rangle$$

Degrees of freedom irrelevant to theory but impactful in systems

- ▶ Choice of codewords
- ▶ Encoding circuit
- ▶ Parity check circuit



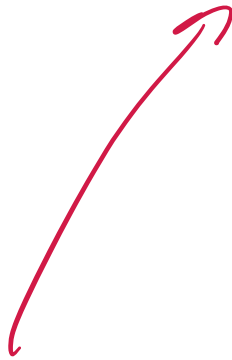
$$\bar{X}|\bar{\psi}\rangle = \bar{X}(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle)$$

$$= I\alpha(|00\rangle + |11\rangle) + \beta(|01\rangle + |10\rangle)$$

$$= (\beta|00\rangle + |11\rangle + \alpha(|01\rangle + |10\rangle))$$

$$= \bar{X}|\bar{\psi}\rangle$$

$ +\rangle$	\longrightarrow	\otimes	$ \psi\rangle$
$s_0 = X$			$s_0 = X$
$n_0 = Z$			$n_0 = I$
$\bar{x}_0 = I$			$\bar{x}_0 = X$
$\bar{z}_0 = Z$			$\bar{z}_0 = Z$



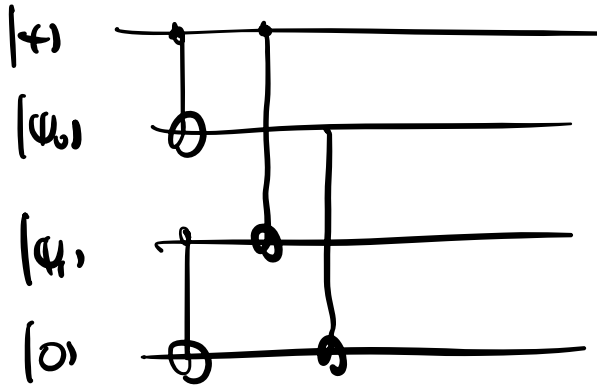
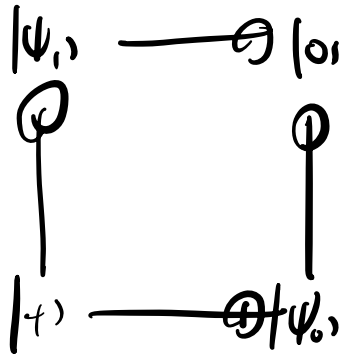
Four-qubit quantum error (detection) codes

$RM(-1, 2)$ [4, 0, ∞]	1	$RM(0, 2)$ [1111] [4, 1, 4]	2	$RM(1, 2)$ [4, 3, 2]	1	$RM(2, 2)$ [4, 4, 1]
$QRM_2(-1, -1)$ [[4, 0, 1]] $S =$ $\{Z_0, Z_1, Z_2, Z_3\}$ codeword= $ 0\rangle 0\rangle 0\rangle 0\rangle$ logicals= \emptyset		$QRM_2(0, 0)$ [[4, 0, 2]] $S =$ $\{Z_0 Z_1, Z_1 Z_2,$ $Z_2 Z_3, Z_3 Z_0$ $X_0 X_1 X_2 X_3\}$ logicals= \emptyset		$QRM_2(1, 1)$ [[4, 0, 2]] $S =$ $\{Z_0 Z_1 Z_2 Z_3,$ $X_0 X_1, X_1 X_2,$ $X_2 X_3, X_3 X_0\}$ logicals= \emptyset		$QRM_2(2, 2)$ [[4, 0, 1]] $S =$ $\{X_0, X_1, X_2, X_3\}$ codeword= $ +\rangle +\rangle +\rangle +\rangle$ logicals= \emptyset
	$QRM_2(-1, 0)$ [[4, 1, 1]] $S =$ $\{Z_0 Z_1, Z_1 Z_2,$ $Z_2 Z_3, Z_3 Z_0\}$ logicals= $\{XXXX, Z, S, T, \dots\}$		$QRM_2(0, 1)$ [[4, 2, 2]] $S =$ $\{Z_0 Z_1 Z_2 Z_3,$ $X_0 X_1 X_2 X_3\}$ logicals= $\{H, ZZ, SSSS, XX, \sqrt{X} \sqrt{X} \sqrt{X} \sqrt{X}\}$		$QRM_2(1, 2)$ [[4, 1, 1]] $S =$ $\{X_0 X_1, X_1 X_2,$ $X_2 X_3, X_3 X_0\}$ logicals= $\{ZZZZ, X, \sqrt{X}, \sqrt[4]{X}, \dots\}$	
		$QRM_2(-1, 1)$ [[4, 3, 1]] $S =$ $\{Z_0 Z_1 Z_2 Z_3\}$ logicals= $\{Z, ZZ, SS, SSSS, TTTT, XX, XXXX\}$		$QRM_2(0, 2)$ [[4, 3, 1]] $S =$ $\{X_0 X_1 X_2 X_3\}$ logicals= $\{ZZ, ZZZZ, X, XX, \sqrt{X} \sqrt{X}, \sqrt{X} \sqrt{X} \sqrt{X} \sqrt{X}, \sqrt[4]{X} \sqrt[4]{X} \sqrt[4]{X} \sqrt[4]{X}\}$		
			$QRM_2(-1, 2)$ [[4, 4, 1]] $S = \{\}$ logicals= $\{H, Z, ZZ, ZZZZ, SSSS, X, XX, XXXX, \sqrt{X} \sqrt{X} \sqrt{X} \sqrt{X}\}$			

$$QRM_{m=2}(q = 0, r = 1) \in [[n = 4, k = 2, d = 2]]$$

The smallest quantum error detecting code

Codes-as-ansatz



Eight-qubit quantum error (correction) codes

$RM(-1, 3)$ [8, 0, ∞]	1	$RM(0, 3)$ [8, 1, 8]	3	$RM(1, 3)$ [8, 4, 4]	3	$RM(2, 3)$ [8, 7, 2]	1	$RM(3, 3)$ [8, 8, 1]	
$QRM_3(-1, -1)$ [[8, 0, 1]] $S : 8Z^{\otimes 1}$ codeword= $ 0\rangle^{\otimes 8}$ logicals= \emptyset		$QRM_3(0, 0)$ [[8, 0, 2]] $S :$ $12Z^{\otimes 2}, 1X^{\otimes 8}$ logicals= \emptyset		$QRM_3(1, 1)$ [[8, 0, 4]] $S :$ $6Z^{\otimes 4}, 6X^{\otimes 4}$ logicals= \emptyset		$QRM_3(2, 2)$ [[8, 0, 2]] $S :$ $1Z^{\otimes 8}, 12X^{\otimes 2}$ logicals= \emptyset		$QRM_3(3, 3)$ [[8, 0, 1]] $S : 8X^{\otimes 1}$ codeword= $ +\rangle^{\otimes 8}$ logicals= \emptyset	
	$QRM_3(-1, 0)$ [[8, 1, 1]] $S : 12Z^{\otimes 2}$ logicals= $\{Z, S, T, \dots\}$		$QRM_3(0, 1)$ [[8, 3, 2]] $S :$ $6Z^{\otimes 4}, 1X^{\otimes 8}$ logicals= $\{Z^{\otimes 2}, S^{\otimes 4}, T^{\otimes 8}\}$		$QRM_3(1, 2)$ [[8, 3, 2]] $S :$ $1Z^{\otimes 8}, 6X^{\otimes 4}$ logicals= $\{Z^{\otimes 4}\}$		$QRM_3(2, 3)$ [[8, 1, 1]] $S : 12X^{\otimes 2}$ logicals= $\{Z^{\otimes 8}\}$		
		$QRM_3(-1, 1)$ [[8, 4, 1]] $S : 6Z^{\otimes 4}$ logicals= $\{Z^{\otimes 1,2}, S^{\otimes 2,4}, T^{\otimes 4,8}\}$		$QRM_3(0, 2)$ [[8, 6, 2]] $S :$ $1Z^{\otimes 8}, 1X^{\otimes 8}$ logicals= $\{H, Z^{\otimes 2}, Z^{\otimes 4}, S^{\otimes 8}\}$		$QRM_3(1, 3)$ [[8, 4, 1]] $S : 6X^{\otimes 4}$ logicals= $\{Z^{\otimes 4}, Z^{\otimes 8}\}$			
			$QRM_3(-1, 2)$ [[8, 7, 1]] $S : 1 \times 8Z$ logicals= $\{Z, Z^{\otimes 2}, Z^{\otimes 4}, S^{\otimes 4}, S^{\otimes 8}\}$		$QRM_3(0, 3)$ [[8, 7, 1]] $S : 1 \times 8X$ logicals= $\{Z^{\otimes 2}, Z^{\otimes 4}, Z^{\otimes 8}\}$				
				$QRM_3(-1, 3)$ [[8, 8, 1]] $S = \{\}$ logicals= $\{H, Z, Z^{\otimes 2}, Z^{\otimes 4}, Z^{\otimes 8}, S^{\otimes 8}\}$					

16-qubit quantum error correction codes

$RM(-1, 4)$ [16, 0, ∞]	1	$RM(0, 4)$ [16, 1, 16]	4	$RM(1, 4)$ [16, 5, 8]	6	$RM(2, 4)$ [16, 11, 4]	4	$RM(3, 4)$ [16, 15, 2]	1	$RM(4, 4)$ [16, 16, 1]
$QRM_4(-1, -1)$ [[16, 0, 1]] S : $16Z^{\otimes 1}$ logicals= \emptyset		$QRM_4(0, 0)$ [[16, 0, 2]] S : $32Z^{\otimes 2}, 1X^{\otimes 16}$ logicals= \emptyset		$QRM_4(1, 1)$ [[16, 0, 4]] S : $24Z^{\otimes 4}, 8X^{\otimes 8}$ logicals= \emptyset		$QRM_4(2, 2)$ [[16, 0, 4]] S : $8Z^{\otimes 8}, 24X^{\otimes 4}$ logicals= \emptyset		$QRM_4(3, 3)$ [[16, 0, 2]] S : $1Z^{\otimes 16}, 32X^{\otimes 2}$ logicals= \emptyset		$QRM_4(4, 4)$ [[16, 0, 1]] S : $16X^{\otimes 1}$ logicals= \emptyset
	$QRM_4(-1, 0)$ [[16, 1, 1]] S : $32Z^{\otimes 2}$ logicals= {Z, S, T, ...}	$QRM_4(0, 1)$ [[16, 4, 2]] S : $24Z^{\otimes 4}, 1X^{\otimes 16}$ logicals= {Z ^{⊗2} , S ^{⊗4} , T ^{⊗8} , $\sqrt{T}^{\otimes 16}$ }		$QRM_4(1, 2)$ [[16, 6, 4]] S : $8Z^{\otimes 8}, 8X^{\otimes 8}$ logicals= {H, Z ^{⊗4} , S ^{⊗16} }		$QRM_4(2, 3)$ [[16, 4, 2]] S : $1Z^{\otimes 16}, 24X^{\otimes 4}$ logicals= {Z ^{⊗8} }		$QRM_4(3, 4)$ [[16, 1, 1]] S : $32X^{\otimes 2}$ logicals= {Z ^{⊗16} }		
		$QRM_4(-1, 1)$ [[16, 5, 1]] logicals= {Z ^{⊗1,2} , S ^{⊗2,4} , T ^{⊗4,8} , $\sqrt{T}^{\otimes 8,16}, \sqrt[4]{T}^{\otimes 16}$ }		$QRM_4(0, 2)$ [[16, 10, 2]] logicals= {Z ^{⊗2} , Z ^{⊗4} , S ^{⊗8} , S ^{⊗16} }		$QRM_4(1, 3)$ [[16, 10, 2]] logicals= {Z ^{⊗4} , Z ^{⊗8} }		$QRM_4(2, 4)$ [[16, 5, 1]] logicals= {Z ^{⊗8} , Z ^{⊗16} }		
		$QRM_4(-1, 2)$ [[16, 11, 1]] logicals= {Z ^{⊗1,2,4} , S ^{⊗4,8,16} , T ^{⊗16} }		$QRM_4(0, 3)$ [[16, 14, 2]] logicals= {H, Z ^{⊗2,4,8} , S ^{⊗16} }		$QRM_4(1, 4)$ [[16, 11, 1]] logicals= {Z ^{⊗4} , Z ^{⊗8} , Z ^{⊗16} }				
			$QRM_4(-1, 3)$ [[16, 15, 1]] logicals= {Z ^{⊗1,2,4,8} , S ^{⊗8,16} }		$QRM_4(0, 4)$ [[16, 15, 1]] logicals= {Z ^{⊗2} , Z ^{⊗4} , Z ^{⊗8} , Z ^{⊗16} }					
				$QRM_4(-1, 4)$ [[16, 16, 1]] logicals= {H, Z, Z ^{⊗2} , Z ^{⊗4} , Z ^{⊗8} , Z ^{⊗16} , S ^{⊗16} }						

$$QRM_{m=3}(q = 1, r = 1) \in [[n = 8, k = 0, d = 4]]$$

Our first encounter with a quantum error correcting code

The geometric construction of quantum Reed-Muller codes

The duality of non-Clifford operations

What about non-Reed-Muller codes?

- ▶ Modified Reed-Muller codes. $[[7, 1, 3]]$, a code with transversal Cliffords.
- ▶ Non-CSS codes. $[[5, 1, 3]]$, the smallest qubit QECC

A periodic table of classical trit generalized Reed-Muller codes

			$GRM_3(-1, 0)$ $[1, 0, \infty]_3$	$GRM_3(0, 0)$ $[1, 1, 1]_3$			
		$GRM_3(-1, 1)$ $[3, 0, \infty]_3$	$GRM_3(0, 1)$ $[3, 1, 3]_3$	$GRM_3(1, 1)$ $[3, 2, 2]_3$	$GRM_3(2, 1)$ $[3, 3, 1]_3$		
	$GRM_3(-1, 2)$ $[9, 0, \infty]_3$	$GRM_3(0, 2)$ $[9, 1, 9]_3$	$GRM_3(1, 2)$ $[9, 3, 6]_3$	$GRM_3(2, 2)$ $[9, 6, 3]_3$	$GRM_3(3, 2)$ $[9, 8, 2]_3$	$GRM_3(4, 2)$ $[9, 9, 1]_3$	
$GRM_3(-1, 3)$ $[27, 0, \infty]_3$	$GRM_3(0, 3)$ $[27, 1, 27]_3$	$GRM_3(1, 3)$ $[27, 4, 18]_3$	$GRM_3(2, 3)$ $[27, 10, 9]_3$	$GRM_3(3, 3)$ $[27, 17, 6]_3$	$GRM_3(4, 3)$ $[27, 23, 3]_3$	$GRM_3(5, 3)$ $[27, 26, 2]_3$	$GRM_3(6, 3)$ $[27, 27, 1]_3$

Trivial codes

Repetition codes

Single parity-check codes

Universe codes

One-qutrit quantum (stabilized) states

$GRM(-1, 0)$ $[1, 0, \infty]_3$	1	$GRM(0, 0)$ $[1, 1, 1]_3$
$QGRM_0^3(-1, -1)$ $[[1, 0, 1]]_3$ $S = \left\{ Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} \right\}$ codeword = $ 0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ logicals = \emptyset		$QGRM_0^3(0, 0)$ $[[1, 0, 1]]_3$ $S = \left\{ X = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}$ codeword = $ +\rangle = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ logicals = \emptyset
	$QGRM_0^3(-1, 0)$ $[[1, 1, 1]]_3$ $S = \{ \}$ codeword = $ \psi\rangle$ logicals = U	

Three-qutrit quantum error (detection) codes

$GRM_3(-1, 1)$ [[3, 0, ∞]] ₃	1	$GRM_3(0, 1)$ [[3, 1, 3]] ₃	1	$GRM_3(1, 1)$ [[3, 2, 2]] ₃	1	$GRM_3(2, 1)$ [[3, 3, 1]] ₃
$QGRM_1^3(-1, -1)$ [[3, 0, 1]] ₃ $S = \{Z_0, Z_1, Z_2\}$ codeword= $ 0\rangle 0\rangle 0\rangle$ logicals= \emptyset		$QGRM_1^3(0, 0)$ [[3, 0, 2]] ₃ $S = \{Z_0 Z_1, Z_1 Z_2, Z_2 Z_0, X_0 X_1 X_2\}$ logicals= \emptyset		$QGRM_1^3(1, 1)$ [[3, 0, 2]] ₃ $S = \{Z_0 Z_1 Z_2, X_0 X_1, X_1 X_2, X_2 X_0\}$ logicals= \emptyset		$QGRM_1^3(2, 2)$ [[3, 0, 1]] ₃ $S = \{X_0, X_1, X_2\}$ codeword= $ +\rangle +\rangle +\rangle$ logicals= \emptyset
	$QGRM_1^3(-1, 0)$ [[3, 1, 1]] ₃ $S = \{Z_0 Z_1, Z_1 Z_2, Z_2 Z_0\}$ logicals= $\{XXX, Z, S, T, \dots\}$		$QGRM_1^3(0, 1)$ [[3, 1, 2]] ₃ $S = \{Z_0 Z_1 Z_2, X_0 X_1 X_2\}$ logicals= $\{H, ZZ, SSS, XX, \sqrt{X}\sqrt{X}\sqrt{X}\}$		$QGRM_1^3(1, 2)$ [[3, 1, 1]] ₃ $S = \{X_0 X_1, X_1 X_2, X_2 X_0\}$ logicals= $\{ZZZ, X, \sqrt{X}, \sqrt[4]{X}, \dots\}$	
		$QGRM_1^3(-1, 1)$ [[3, 2, 1]] ₃ $S = \{Z_0 Z_1 Z_2\}$ logicals= $\{Z, ZZ, SS, SSS, XX, XXX\}$		$QGRM_1^3(0, 2)$ [[3, 2, 1]] ₃ $S = \{X_0 X_1 X_2\}$ logicals= $\{ZZ, ZZZ, X, XX, \sqrt{X}\sqrt{X}, \sqrt{X}\sqrt{X}\sqrt{X}\}$		
			$QGRM_1^3(-1, 2)$ [[3, 3, 1]] ₃ $S = \{\}$ logicals= $\{H, Z, ZZ, ZZZ, SSS, X, XX, XXX, \sqrt{X}\sqrt{X}\sqrt{X}\}$			

Nine-qutrit quantum error (correction) codes

$GRM_3(-1, 2)$ [9, 0, ∞] ₃	1	$GRM_3(0, 2)$ [9, 1, 9] ₃	2	$GRM_3(1, 2)$ [9, 3, 6] ₃	3	$GRM_3(2, 2)$ [9, 6, 3] ₃	2	$GRM_3(3, 2)$ [9, 8, 2] ₃	1	$GRM_3(4, 2)$ [9, 9, 1] ₃
$QGRM_2^3(-1, -1)$ [[9, 0, 1]] ₃ $S =$ $9Z^{\otimes 1}$ logicals= \emptyset		$QGRM_2^3(0, 0)$ [[9, 0, 2]] ₃ $S =$ $?Z^{\otimes 2}, 1X^{\otimes 9}$ logicals= \emptyset		$QGRM_2^3(1, 1)$ [[9, 0, 3]] ₃ $S =$ $6Z^{\otimes 3}, ?X^{\otimes 6}$ logicals= \emptyset		$QGRM_2^3(2, 2)$ [[9, 0, 3]] ₃ $S =$ $?Z^{\otimes 6}, 6X^{\otimes 3}$ logicals= \emptyset		$QGRM_2^3(3, 3)$ [[9, 0, 2]] ₃ $S =$ $1Z^{\otimes 9}, ?X^{\otimes 2}$ logicals= \emptyset		$QGRM_2^3(4, 4)$ [[9, 0, 1]] ₃ $S =$ $9X^{\otimes 1}$ logicals= \emptyset
$QGRM_2^3(-1, 0)$ [[9, 1, 1]] ₃ $S =$ $?Z^{\otimes 2}$ logicals= {Z, S, T, ...}		$QGRM_2^3(0, 1)$ [[9, 2, 2]] ₃ $S =$ $6Z^{\otimes 3}, 1X^{\otimes 9}$ logicals= {Z ^{⊗2} , S ^{⊗3} , T ^{⊗9} }		$QGRM_2^3(1, 2)$ [[9, 3, 3]] ₃ $S =$ $?Z^{\otimes 6}, ?X^{\otimes 6}$ logicals= {H, Z ^{⊗3} , S ^{⊗9} }		$QGRM_2^3(2, 3)$ [[9, 2, 2]] ₃ $S =$ $1Z^{\otimes 9}, 6X^{\otimes 3}$ logicals= {Z ^{⊗6} }		$QGRM_2^3(3, 4)$ [[9, 1, 1]] ₃ $S =$ $?X^{\otimes 2}$ logicals= {Z ^{⊗9} }		
		$QGRM_2^3(-1, 1)$ [[9, 3, 1]] ₃ logicals= {Z ^{⊗1,2} , S ^{⊗2,3} , T ^{⊗6,9} }		$QGRM_2^3(0, 2)$ [[9, 5, 2]] ₃ logicals= {Z ^{⊗2,3} , S ^{⊗6,9} }		$QGRM_2^3(1, 3)$ [[9, 5, 2]] ₃ logicals= {Z ^{⊗3,6} }		$QGRM_2^3(2, 4)$ [[9, 3, 1]] ₃ logicals= {Z ^{⊗6,9} }		
		$QGRM_2^3(-1, 2)$ [[9, 6, 1]] ₃ logicals= {Z ^{⊗1,2,3} , S ^{⊗3,6,9} }		$QGRM_2^3(0, 3)$ [[9, 7, 2]] ₃ logicals= {H, Z ^{⊗2,3,6} , S ^{⊗9} }		$QGRM_2^3(1, 4)$ [[9, 6, 1]] ₃ logicals= {Z ^{⊗3,6,9} }				
				$QGRM_2^3(-1, 3)$ [[9, 8, 1]] ₃ logicals= {Z ^{⊗1,2,3,6} , S ^{⊗6,9} }		$QGRM_2^3(0, 4)$ [[9, 8, 1]] ₃ logicals= {Z ^{⊗2,3,6,9} }				
				$QGRM_2^3(-1, 4)$ [[9, 9, 1]] ₃ logicals= {H, Z ^{⊗1,2,3,6,9} , S ^{⊗9} }						

A periodic table of classical quart generalized Reed-Muller codes

				$GRM_4(-1, 0)$ $[1, 0, \infty]_4$	$GRM_4(0, 0)$ $[1, 1, 1]_4$			
		$GRM_4(-1, 1)$ $[4, 0, \infty]_4$	$GRM_4(0, 1)$ $[4, 1, 4]_4$	$GRM_4(1, 1)$ $[4, 2, 3]_4$	$GRM_4(2, 1)$ $[4, 3, 2]_4$	$GRM_4(3, 1)$ $[4, 4, 1]_4$		
$GRM_4(-1, 2)$ $[16, 0, \infty]_4$	$GRM_4(0, 2)$ $[16, 1, 16]_4$	$GRM_4(1, 2)$ $[16, 3, 12]_4$	$GRM_4(2, 2)$ $[16, 6, 8]_4$	$GRM_4(3, 2)$ $[16, 10, 4]_4$	$GRM_4(4, 2)$ $[16, 13, 3]_4$	$GRM_4(5, 2)$ $[16, 15, 2]_4$	$GRM_4(6, 2)$ $[16, 16, 1]_4$	

Trivial codes

Repetition codes

Self-dual code

Single parity-check codes

Universe codes