A periodic table of quantum error correction codes

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2026

NISQ vs. FTQC

noisy intermediate scale quantum

10, 100, 1000 qubits

fault-tolerant QC

quantum error correcting codes (QECC)

NISQ

- past ~10 years

  random sampling

  optimization

  chemistry

FTQC

- 1M qubits

  phys qubits ~0.1% error

  ~1% error

  logical qubits 1/18 error

  next ? years

Quantum error correction codes: theory and practical systems

Theory cares about

- Code taxonomy
- Code parameters
- Code specification

Practical systems need

- Code words
- Encoding circuit, and its compilation
- Logical operators, and their compilation
- Error syndrome parity check circuit, and its compilation
- Decoding algorithm

What this talk is about

- A tutorial on quantum Reed-Muller codes, a taxonomy where the existence of logical operators is the most clear
- A geometric approach to establishing what operators are available, and its impact on compilation
- Stabilizers and codewords of quantum Reed-Muller codes, and their potential as symmetry-preserving quantum ansatzes.
- A correct generalization of the geometric view of quantum Reed-Muller codes to qudits.

A periodic table of classical Reed-Muller codes

$RM(-1,0)$ $[1,0,\infty]$	1	$RM(0,0)$ $[1,1,1]$										
$RM(-1,1)$ $[2,0,\infty]$	1	$RM(0,1)$ $[2,1,2]$	1	$RM(1,1)$ $[2,2,1]$								
$RM(-1,2)$ $[4,0,\infty]$	1	$RM(0,2)$ $[4,1,4]$	2	$RM(1,2)$ $[4,3,2]$	1	$RM(2,2)$ $[4,4,1]$						
$RM(-1,3)$ $[8,0,\infty]$	1	$RM(0,3)$ $[8,1,8]$	3	$RM(1,3)$ $[8,4,4]$	3	$RM(2,3)$ $[8,7,2]$	1	$RM(3,3)$ $[8,8,1]$				
$RM(-1,4)$ $[16,0,\infty]$	1	$RM(0,4)$ $[16,1,16]$	4	$RM(1,4)$ $[16,5,8]$	6	$RM(2,4)$ $[16,11,4]$	4	$RM(3,4)$ $[16,15,2]$	1	$RM(4,4)$ $[16,16,1]$		
$RM(-1,5)$ $[32,0,\infty]$	1	$RM(0,5)$ $[32,1,32]$	5	$RM(1,5)$ $[32,6,16]$	10	$RM(2,5)$ $[32,16,8]$	10	$RM(3,5)$ $[32,26,4]$	5	$RM(4,5)$ $[32,31,2]$	1	$RM(5,5)$ $[32,32,1]$

Trivial codes

Repetition codes

Punctured Hadamard codes

Self-dual codes

Extended H

Single parit

Universe c

$RM(r = 0, m = 0)$

$x: \{[0], [1]\}$

$y: \{[0], [1]\}$

A single classical bit

$y = Gx$

- Messages x
- Codewords y
- Generator matrix $G = [1]$

A $[n = 1, k = 1, d = 1]$ code

- n
- k
- d

$RM(r = 1, m = 1) \in [n = 2, k = 2, d = 1]$

$2^m = n$

Two classical bits

$y = Gx$

- Messages $x \in \{[0,0],[0,1],[1,0],[1,1]\}$
- Codewords $y \in \left\{ \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$
- Generator matrix $G = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$m \{ \alpha_0$

$G = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$

A $[n = 2, k = 2, d = 1]$ code

- n
- k
- d

$$G = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad y = x \cdot G$$

$$x \quad \cdot \quad G \qquad\qquad y$$

$$[0, 0] \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = [0, 0]$$

$$[0, 1] \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = [0, 1]$$

$$[1, 0] \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = [1, 1]$$

$$[1, 1] \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = [1, 0]$$

$RM(r=0, m=1) \in [n=2, k=1, d=2]$

$y = Gx$

- Messages $x \in \{[0], [1]\}$
- Codewords $y \in \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$
- Generator matrix $G = \left[\begin{array}{c} 1 \\ \hline 1 \end{array}\right]$, the message and parity check bits

$Hy \equiv 0 \mod 2$

- Indication that no error was detected
- Parity check matrix $H = \begin{bmatrix} 1 & 1 \end{bmatrix}$

The only single parity-check, repetition code

- $G = H^\top$, so it is also self-dual

$RM(r=0, m=1)$

$f(x_0) = x_0^0$

$$G = \begin{array}{c} n \\ k\left[\begin{matrix} 1 & 1 \end{matrix}\right] \end{array} \qquad H^T = \left[\begin{matrix} 1 & 1 \end{matrix}\right]$$

$p = qG$

$q = [0] \qquad [0][1\ 1] = [0\ 0]$

$q = [1] \qquad [1][1\ 1] = [1\ 1]$

$n = 2 \qquad n = 2^m$

$k = 1$

$d = 2$

$$RM(r=1, m=2) : [4,3,2]$$

$$n = 2^m = 2^2 = 4$$

$$G = \begin{array}{c} 4 \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{l} f(x_0,x_1) = 1 \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} f(x_0,x_1) \begin{array}{l} x_0 \\ x_1 \end{array} \end{array}$$

$$[0\ 0\ 0]\,G = [0\ 0\ 0\ 0]$$

$$[0\ 0\ 1]\,G = [0\ 1\ 0\ 1]$$

$$[0\ 1\ 0]\,G = [0\ 0\ 1\ 1]$$

$$[0\ 1\ 1]\,G = [0\ 1\ 1\ 0]$$

$$[1\ 0\ 0]\,G =$$

$$[1\ 0\ 1]\,G =$$

$$[1\ 1\ 0]\,G =$$

$$[1\ 1\ 1]\,G =$$

$RM(r = 1, m = 3) \in [n = 8, k = 4, d = 4]$

Usually, we start with this code

- Smallest non-repetition error correcting code
- Is an extended Hamming code
- Is a truncated Hadamard code
- Closely related $[7, 4, 3]$ saturates the Hamming bound.

$RM(r=1, m=3) = [8, 4, 4]$

$n = 2^m = 2^3 = 8$

$$G = k \begin{array}{c} \overset{8}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}} \end{array} \begin{array}{l} f(x_0, x_1, x_2) \quad 3C0 \\ \left.\begin{array}{l} f() = x_0 \\ f() = x_1 \\ f() = x_2 \end{array}\right\} 3C1 \end{array}$$

$G = H^T$

H_q

Reed-Solomon

Turbo codes

LDPC

3-qbit repetition code

9-qubit Shor QECC

5-qubit Perfect code

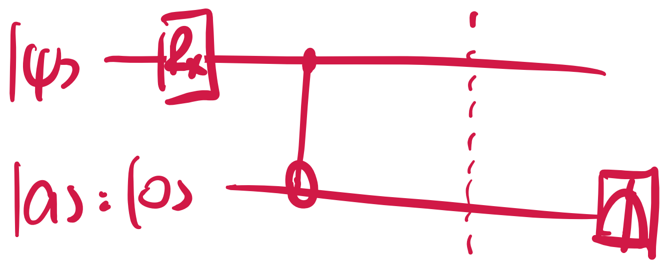
An error model

$$R_x(\theta) = \cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right) X$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$R_x(\theta)|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|\psi\rangle - i\sin\left(\frac{\theta}{2}\right) X|\psi\rangle$$

$$\cos\left(\frac{\theta}{2}\right)\left[|\psi\rangle \otimes |0\rangle\right] - i\sin\left(\frac{\theta}{2}\right)\left[X|\psi\rangle \otimes |1\rangle\right]$$



$$|\psi\rangle \otimes |0\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle$$

$$\xrightarrow{\text{CNOT}} \alpha|00\rangle + \beta|11\rangle$$

Stabilized States

Logical states

Codes

→ encoding

→ logical operations

→ syndrome extraction

Fundamental differences between ECC and QECC

No-cloning theorem

There is no way to duplicate an arbitrary quantum state. Suppose a cloning operation U_c exists. Then:

- $$U_c(|\phi\rangle \otimes |\omega\rangle) = |\phi\rangle \otimes |\phi\rangle \,,$$
  $$U_c(|\psi\rangle \otimes |\omega\rangle) = |\psi\rangle \otimes |\psi\rangle \,,$$
  for arbitrary states $|\phi\rangle \,, |\psi\rangle$ we wish to copy.
- The overlap of the final states is:
  $$\langle\phi| \otimes \langle\phi| \, |\psi\rangle \otimes |\psi\rangle = \langle\phi| \, |\psi\rangle \cdot \langle\phi| \, |\psi\rangle = (\langle\phi| \, |\psi\rangle)^2$$
- The overlap of the final states is also:
  $$\langle\phi| \otimes \langle\phi| \, |\psi\rangle \otimes |\psi\rangle = \langle\phi| \otimes \langle\omega| \, U^\dagger U \, |\psi\rangle \otimes |\omega\rangle = \langle\phi| \otimes \langle\omega| \, |\psi\rangle \otimes |\omega\rangle = \langle\phi| \, |\psi\rangle$$
- $(\langle\phi| \, |\psi\rangle)^2 = \langle\phi| \, |\psi\rangle$, so $\langle\phi| \, |\psi\rangle = 0$, or $\langle\phi| \, |\psi\rangle = 1$, $|\phi\rangle$ and $|\psi\rangle$ cannot be arbitrary states as claimed.

One-qubit quantum (stabilized) states

$RM(-1,0)$ $[1,0,\infty]$	1	$RM(0,0)$ $[1,1,1]$
$QRM_0(-1,-1)$ $[[1,0,1]]$ $S=\{Z\}$ codeword$=\|0\rangle$ logicals$=\emptyset$		$QRM_0(0,0)$ $[[1,0,1]]$ $S=\{X\}$ codeword$=\|+\rangle$ logicals$=\emptyset$
	$QRM_0(-1,0)$ $[[1,1,1]]$ $S=\{\}$ codeword$=\|\psi\rangle$ logicals$=U$	

Two-qubit quantum error "codes"

$[1\ 1]$

$RM(-1,1)$ $[2,0,\infty]$	1	$RM(0,1)$ $[2,1,2]$	1	$RM(1,1)$ $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ $[2,2,1]$
$QRM_1(-1,-1)$ $[[2,0,1]]$ $S=\{Z_0,Z_1\}$ ZI IZ codeword= $\|0\rangle\|0\rangle$ logicals= $\emptyset$		$QRM_1(0,0)$ $[[2,0,2]]$ $S=$ $\{Z_0Z_1, X_0X_1\}$ codeword= $\|\Phi^+\rangle$ logicals= $\emptyset$		$QRM_1(1,1)$ $[[2,0,1]]$ $S=\{X_0,X_1\}$ codeword= $\|+\rangle\|+\rangle$ logicals= $\emptyset$
	$QRM_1(-1,0)$ $[[2,1,1]]$ $S=\{Z_0Z_1\}$ logicals= $\{XX,Z,S,T,\ldots\}$		$QRM_1(0,1)$ $[[2,1,1]]$ $S=\{X_0X_1\}$ logicals= $\{ZZ,X,\sqrt{X},\sqrt[4]{X},\ldots\}$	
		$QRM_1(-1,1)$ $[[2,2,1]]$ $S=\{\}$ logicals= $\{H,Z,ZZ,SS,X,XX,\sqrt{X}\sqrt{X}\}$		

$QRM_{m=1}(q=-1, r=0) \in [[n=2, k=1, d=1]]$

$Z \otimes Z \, |00\rangle$

$ZZ \, |11\rangle$

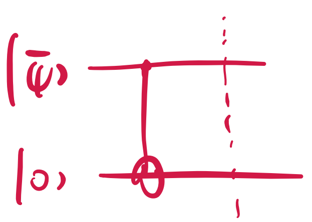
$ZZ: \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}$

$|\bar{\psi}\rangle = \alpha|\bar{0}\rangle + \beta|\bar{1}\rangle$

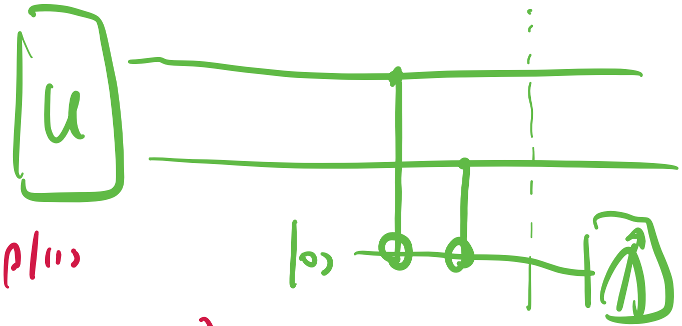
$= \alpha|00\rangle + \beta|11\rangle$

A first attempt at making a QECC

- The code's stabilizer generators
- A valid choice of codewords
- Encoding circuit
- Parity check error syndrome extraction circuit
- logical operations

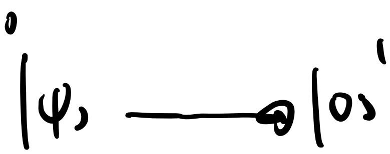


$\alpha|00\rangle + \beta|11\rangle$



$$\bar{X}|\bar{\psi}\rangle = \bar{X}\left(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle\right)$$
$$= XX\left(\alpha|00\rangle + \beta|11\rangle\right)$$
$$= \beta|00\rangle + \alpha|11\rangle$$
$$= \bar{X}|\bar{\psi}\rangle$$

$$\alpha|00\rangle\otimes|0\rangle + \beta|11\rangle\otimes|0\rangle$$
$$+ \gamma|01\rangle\otimes|1\rangle + \delta|10\rangle\otimes|1\rangle$$



	X_0	X_1	Z_0	Z_1
$\bar{X}_0$	1	1		
n_0		1		
$\bar{Z}_0$			1	
S_0			1	1

$$\bar{Z}|\bar{\psi}\rangle = \bar{Z}\left(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle\right)$$
$$= ZI\left(\alpha|00\rangle + \beta|11\rangle\right)$$
$$= \alpha|00\rangle - \beta|11\rangle$$
$$= \bar{Z}|\bar{\psi}\rangle$$

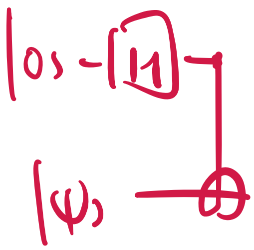
$QRM_{m=1}(q = 0, r = 1) \in [[n = 2, k = 1, d = 1]]$

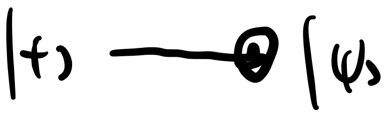
A second attempt at making a QECC

- The code's stabilizer generators $XX : \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$
- A valid choice of codewords $\rightarrow$ $|\bar{0}\rangle : |00\rangle + |11\rangle$ $\quad |\bar{1}\rangle : |01\rangle + |10\rangle$
- Encoding circuit
- Parity check error syndrome extraction circuit

Degrees of freedom irrelevant to theory but impactful in systems

- Choice of codewords
- Encoding circuit
- Parity check circuit





$S_0 : X$

$n_0 : Z$

$\bar{X}_0 : I$

$\bar{Z}_0 : Z$

$S_0 : X$

$n_0 : I$

$\bar{X}_0 : X$

$\bar{Z}_0 : Z$

$$\bar{X}|\bar{\psi}\rangle = \bar{X}(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle)$$

$$= IX(\alpha(|00\rangle + |11\rangle) + \beta(|01\rangle + |10\rangle))$$

$$= (\beta(|00\rangle + |11\rangle) + \alpha(|01\rangle + |10\rangle))$$

$$= \bar{X}|\bar{\psi}\rangle$$

Four-qubit quantum error (detection) codes

$RM(-1,2)$ $[4,0,\infty]$	1	$RM(0,2)$ $[4,1,4]$	2	$RM(1,2)$ $[4,3,2]$	1	$RM(2,2)$ $[4,4,1]$
$QRM_2(-1,-1)$ $[[4,0,1]]$ $S=$ $\{Z_0,Z_1,Z_2,Z_3\}$ codeword= $\|0\rangle\,\|0\rangle\,\|0\rangle\,\|0\rangle$ logicals= $\emptyset$		$QRM_2(0,0)$ $[[4,0,2]]$ $S=$ $\{Z_0Z_1,Z_1Z_2,$ Z_2Z_3,Z_3Z_0 $X_0X_1X_2X_3\}$ logicals= $\emptyset$		$QRM_2(1,1)$ $[[4,0,2]]$ $S=$ $\{Z_0Z_1Z_2Z_3,$ $X_0X_1,X_1X_2,$ $X_2X_3,X_3X_0\}$ logicals= $\emptyset$		$QRM_2(2,2)$ $[[4,0,1]]$ $S=$ $\{X_0,X_1,X_2,X_3\}$ codeword= $\|+\rangle\,\|+\rangle\,\|+\rangle\,\|+\rangle$ logicals= $\emptyset$
	$QRM_2(-1,0)$ $[[4,1,1]]$ $S=$ $\{Z_0Z_1,Z_1Z_2,$ $Z_2Z_3,Z_3Z_0\}$ logicals= $\{XXXX,Z,S,T,\ldots\}$		$QRM_2(0,1)$ $[[4,2,2]]$ $S=$ $\{Z_0Z_1Z_2Z_3,$ $X_0X_1X_2X_3\}$ logicals= $\{H,ZZ,SSSS,XX,\sqrt{X}\sqrt{X}\sqrt{X}\sqrt{X}\}$		$QRM_2(1,2)$ $[[4,1,1]]$ $S=$ $\{X_0X_1,X_1X_2,$ $X_2X_3,X_3X_0\}$ logicals= $\{ZZZZ,X,\sqrt{X},\sqrt[4]{X},\ldots\}$	
		$QRM_2(-1,1)$ $[[4,3,1]]$ $S=$ $\{Z_0Z_1Z_2Z_3\}$ logicals= $\{Z,ZZ,SS,SSSS,TTTT,XX,XXXX\}$		$QRM_2(0,2)$ $[[4,3,1]]$ $S=$ $\{X_0X_1X_2X_3\}$ logicals= $\{ZZ,ZZZZ,X,XX,\sqrt{X}\sqrt{X},\sqrt{X}\sqrt{X}\sqrt{X}\sqrt{X},\sqrt[4]{X}\sqrt[4]{X}\sqrt[4]{X}\sqrt[4]{X}\}$		
			$QRM_2(-1,2)$ $[[4,4,1]]$ $S=\{\}$ logicals= $\{H,Z,ZZ,ZZZZ,SSSS,X,XX,XXXX,\sqrt{X}\sqrt{X}\sqrt{X}\sqrt{X}\}$			

$QRM_{m=2}(q = 0, r = 1) \in [[n = 4, k = 2, d = 2]]$

The smallest quantum error detecting code

Codes-as-ansatz

[[4,2,2]]

quantum hypercube code:

$\{ [[2,1,1]],\ [[4,2,2]],\ [[8,3,2]],\ [[16,4,2]] \}$

A: phantom codes

: relabeling $\underline{\text{qubits}}$ $\longrightarrow$ complete CNOTs

B: $\sqrt[n]{I}^{\otimes n} \longrightarrow \overline{gCCZ}$

$[[2,1,1]]:\ ZZ^\dagger \longrightarrow \overline{Z}$

$[[4,2,2]]:\ SS^\dagger S^\dagger S \to \overline{CZ}$

$[[8,3,2]]:\ TTTTTTTT \to \overline{CCZ}:\ \delta\left\{\overbrace{\begin{pmatrix}1 & & \\ & \ddots & \\ & & -1\end{pmatrix}}^{\delta}\right.$

quantum tesseract code

$\{ [[1,1,1]],\ [[4,2,2]],\ [[16,6,4]],\ [[64,20,8]] \cdots \}$

$[[256,70,16]]$

A: quasi-phantom.

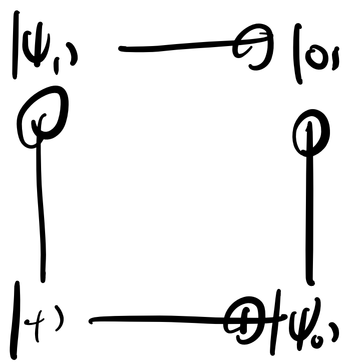
relabeling qubits $\Rightarrow$ nearly complete CNOTs

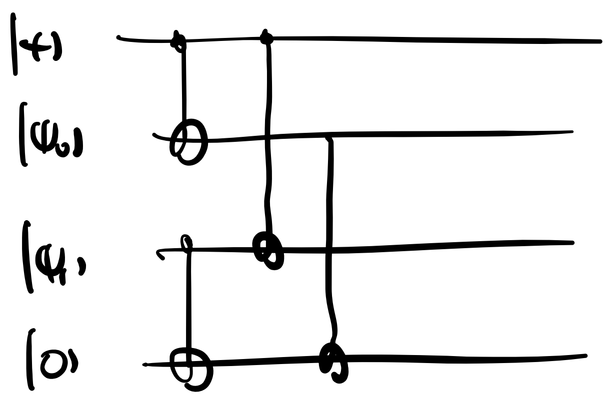
B: $S^{\otimes n} \to \overline{CZ}^{\otimes \frac{k}{2}}$

C: $H^{\otimes n} \to \overline{H}^{\otimes k}$ + SWAPs

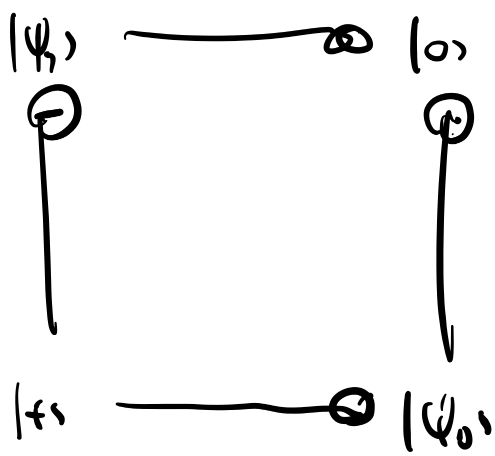
quantum iceberg code

$\{ [[2,0,2]],\ [[4,2,2]],\ [[8,6,2]],\ [[16,14,2]] \cdots \}$





S_0 X
n_0
$\bar{X}_0$
$\bar{Z}_0$
$\bar{X}_1 : X$
$\bar{Z}_1 : Z$
$S_1 : Z$
$n_1 :$



S_0 X
n_0
$\bar{X}$ X
$\bar{Z}_0$
$\bar{X}_1 : X$
$\bar{Z}_1 :$
$S_1 : Z$
$n_1 : X$

$S_1 = X$
$n = Z$
$\bar{X}_0 :$
$\bar{Z}_0 : Z$
$\bar{X}_1$
$\bar{Z}_1$ Z
S_1 Z

$S_0 : X$
$n_0 :$
$\bar{X}_0 : X$
$\bar{Z}_0 : Z$
$\bar{X}_1$
$\bar{Z}_1$
S_1 Z
n_1

$$[[2,1,1]]\ XX = \begin{cases} |\bar{0}\rangle = \begin{bmatrix}1\\0\\0\\1\end{bmatrix} = |00\rangle + |11\rangle \\ |\bar{1}\rangle = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} = |01\rangle + |10\rangle \end{cases}$$

$$[[2,1,1]]\ ZZ = \begin{cases} |\bar{0}\rangle = |00\rangle \\ |\bar{1}\rangle = |11\rangle \end{cases}$$

$$[[4,2,2]] : \begin{cases} |\bar{0}\bar{0}\rangle = (|00\rangle + |11\rangle) \otimes |00\rangle \xrightarrow{\text{Cnot}} |0000\rangle + |1111\rangle \\ |\bar{0}\bar{1}\rangle = (|00\rangle + |11\rangle) \otimes |11\rangle \xrightarrow{\text{Cnot}} |0011\rangle + |1100\rangle \\ |\bar{1}\bar{0}\rangle = (|01\rangle + |10\rangle) \otimes |00\rangle \xrightarrow{\text{Cnot}} |0101\rangle + |1010\rangle \\ |\bar{1}\bar{1}\rangle = (|01\rangle + |10\rangle) \otimes |11\rangle \xrightarrow{\text{Cnot}} |0110\rangle + |1001\rangle \end{cases}$$

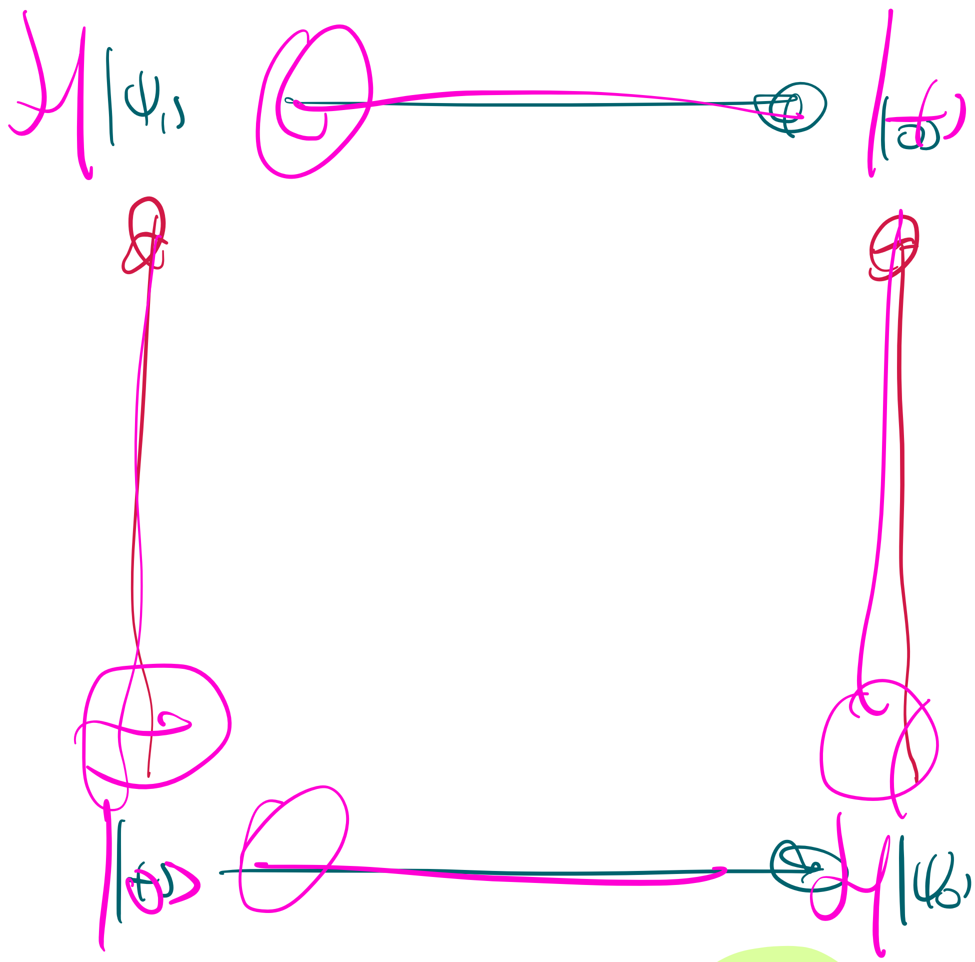
$$\sigma_z = \begin{bmatrix}1&&&\\&1&&\\&&1&\\&&&-1\end{bmatrix}$$

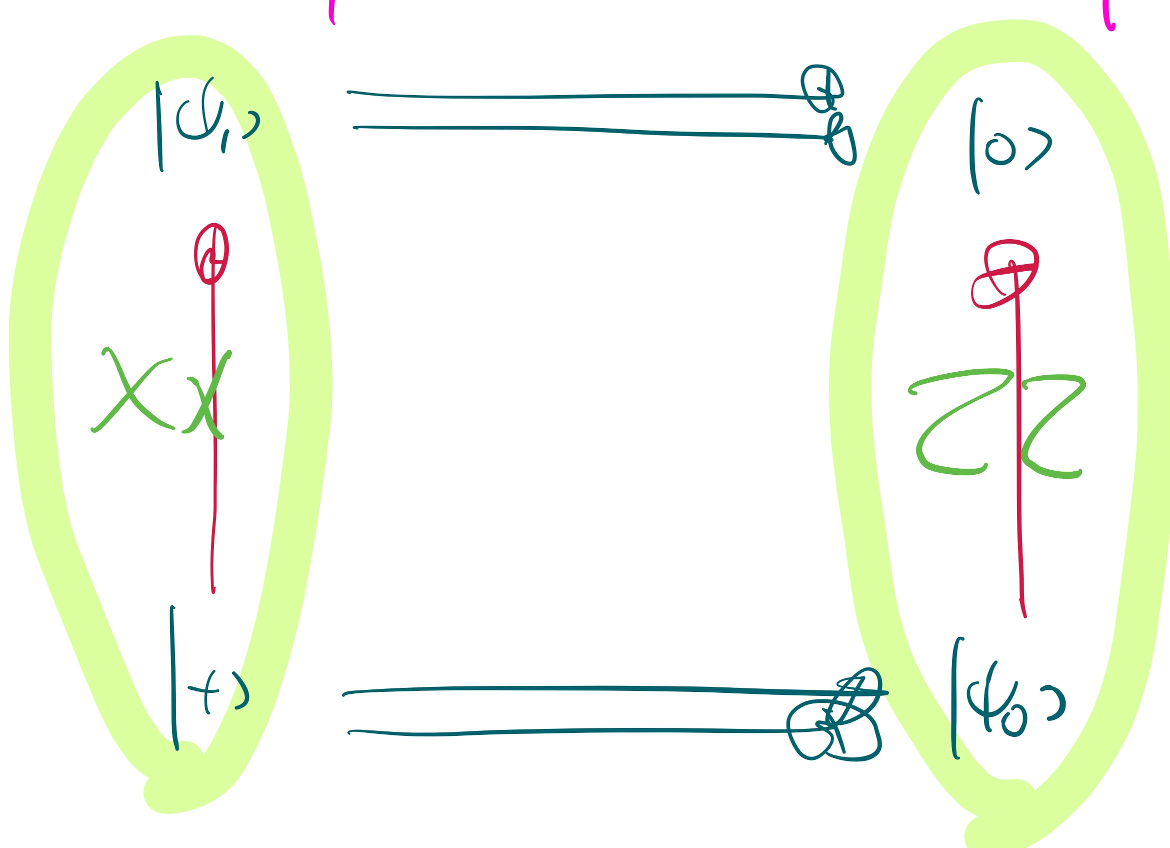
$$SSS^{\dagger}S^{\dagger}$$

$$S = \begin{bmatrix}1&\\&i\end{bmatrix}$$

$$S|1\rangle = i|1\rangle$$

$$SSS^{\dagger}S^{\dagger} = \begin{bmatrix}1&\\&i\end{bmatrix}\begin{bmatrix}1&\\&-i\end{bmatrix}\begin{bmatrix}1&\\&i\end{bmatrix}\begin{bmatrix}1&\\&i\end{bmatrix}$$





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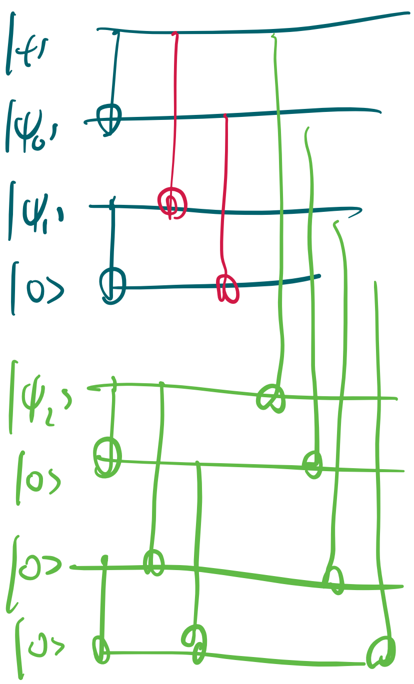
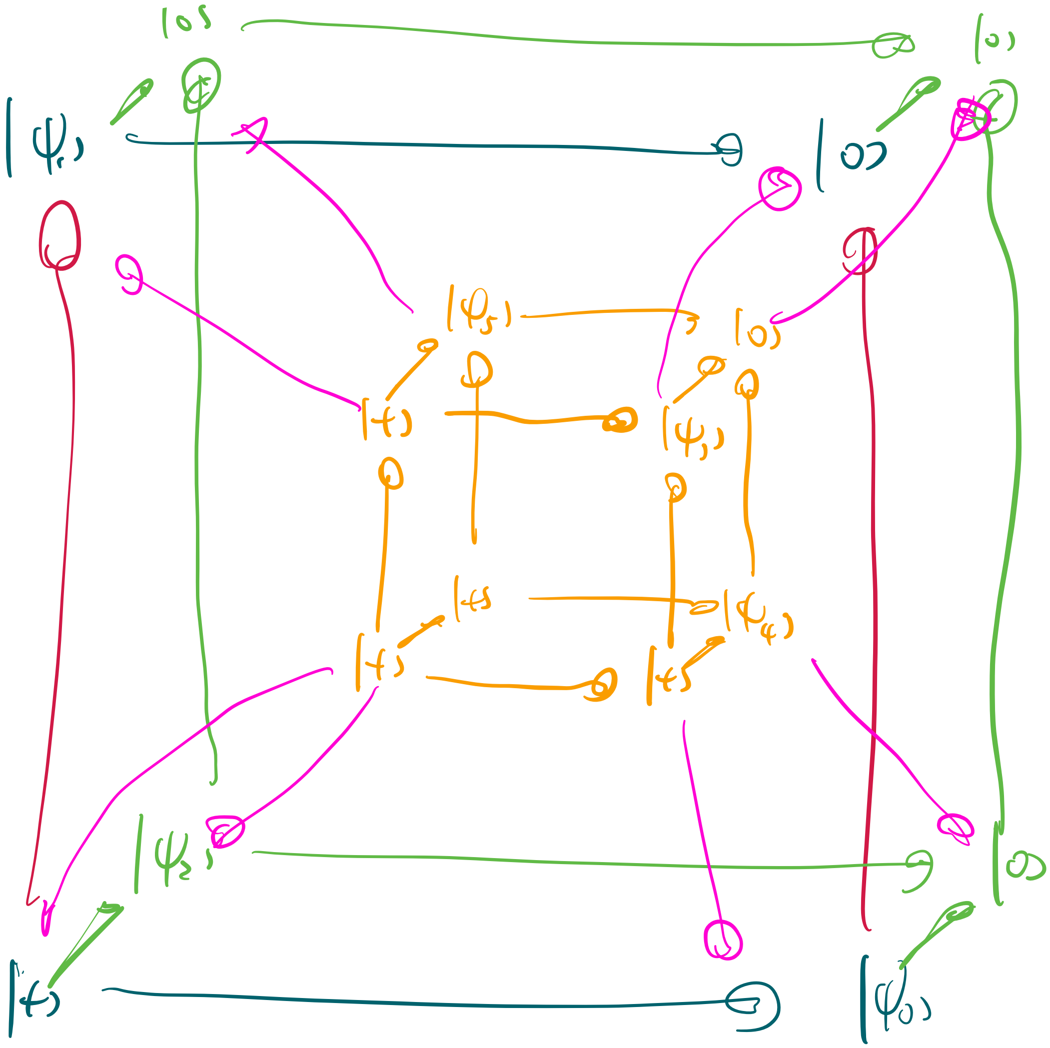
Eight-qubit quantum error (correction) codes

$RM(-1,3)$ $[8,0,\infty]$	1	$RM(0,3)$ $[8,1,8]$	3	$RM(1,3)$ $[8,4,4]$	3	$RM(2,3)$ $[8,7,2]$	1	$RM(3,3)$ $[8,8,1]$
$QRM_3(-1,-1)$ $[[8,0,1]]$ $S: 8Z^{\otimes 1}$ codeword= $\|0\rangle^{\otimes 8}$ logicals= $\emptyset$		$QRM_3(0,0)$ $[[8,0,2]]$ $S:$ $12Z^{\otimes 2}, 1X^{\otimes 8}$ logicals= $\emptyset$		$QRM_3(1,1)$ $[[8,0,4]]$ $S:$ $6Z^{\otimes 4}, 6X^{\otimes 4}$ logicals= $\emptyset$		$QRM_3(2,2)$ $[[8,0,2]]$ $S:$ $1Z^{\otimes 8}, 12X^{\otimes 2}$ logicals= $\emptyset$		$QRM_3(3,3)$ $[[8,0,1]]$ $S: 8X^{\otimes 1}$ codeword= $\|+\rangle^{\otimes 8}$ logicals= $\emptyset$
	$QRM_3(-1,0)$ $[[8,1,1]]$ $S: 12Z^{\otimes 2}$ logicals= $\{Z, S, T, \ldots\}$		$QRM_3(0,1)$ $[[8,3,2]]$ $S:$ $6Z^{\otimes 4}, 1X^{\otimes 8}$ logicals= $\{Z^{\otimes 2}, S^{\otimes 4}, T^{\otimes 8}\}$		$QRM_3(1,2)$ $[[8,3,2]]$ $S:$ $1Z^{\otimes 8}, 6X^{\otimes 4}$ logicals= $\{Z^{\otimes 4}\}$		$QRM_3(2,3)$ $[[8,1,1]]$ $S: 12X^{\otimes 2}$ logicals= $\{Z^{\otimes 8}\}$	
		$QRM_3(-1,1)$ $[[8,4,1]]$ $S: 6Z^{\otimes 4}$ logicals= $\{Z^{\otimes 1,2}, S^{\otimes 2,4}, T^{\otimes 4,8}\}$		$QRM_3(0,2)$ $[[8,6,2]]$ $S:$ $1Z^{\otimes 8}, 1X^{\otimes 8}$ logicals= $\{H, Z^{\otimes 2}, Z^{\otimes 4}, S^{\otimes 8}\}$		$QRM_3(1,3)$ $[[8,4,1]]$ $S: 6X^{\otimes 4}$ logicals= $\{Z^{\otimes 4}, Z^{\otimes 8}\}$		
			$QRM_3(-1,2)$ $[[8,7,1]]$ $S: 1 \times 8Z$ logicals= $\{Z, Z^{\otimes 2}, Z^{\otimes 4}, S^{\otimes 4}, S^{\otimes 8}\}$		$QRM_3(0,3)$ $[[8,7,1]]$ $S: 1 \times 8X$ logicals= $\{Z^{\otimes 2}, Z^{\otimes 4}, Z^{\otimes 8}\}$			
				$QRM_3(-1,3)$ $[[8,8,1]]$ $S = \{\}$ logicals= $\{H, Z, Z^{\otimes 2}, Z^{\otimes 4}, Z^{\otimes 8}, S^{\otimes 8}\}$				


$|+\rangle$
$|\psi_0\rangle$
$|\psi_1\rangle$
$|0\rangle$
$|\psi_2\rangle$
$|0\rangle$
$|0\rangle$
$|0\rangle$


$|0\rangle$
$|0\rangle$
$|\psi_1\rangle$
$|0\rangle$
$|\psi_5\rangle$
$|0\rangle$
$|+\rangle$
$|\psi_3\rangle$
$|+\rangle$
$|\psi_4\rangle$
$|+\rangle$
$|+\rangle$
$|\psi_2\rangle$
$|0\rangle$
$|+\rangle$
$|\psi_0\rangle$

16-qubit quantum error correction codes

$RM(-1,4)$ $[16,0,\infty]$	1	$RM(0,4)$ $[16,1,16]$	4	$RM(1,4)$ $[16,5,8]$	6	$RM(2,4)$ $[16,11,4]$	4	$RM(3,4)$ $[16,15,2]$	1	$RM(4,4)$ $[16,16,1]$

$QRM_4(-1,-1)$ $[[16,0,1]]$ S : $16Z^{\otimes 1}$ logicals= $\emptyset$

$QRM_4(0,0)$ $[[16,0,2]]$ S : $32Z^{\otimes 2}, 1X^{\otimes 16}$ logicals= $\emptyset$

$QRM_4(1,1)$ $[[16,0,4]]$ S : $24Z^{\otimes 4}, 8X^{\otimes 8}$ logicals= $\emptyset$

$QRM_4(2,2)$ $[[16,0,4]]$ S : $8Z^{\otimes 8}, 24X^{\otimes 4}$ logicals= $\emptyset$

$QRM_4(3,3)$ $[[16,0,2]]$ S : $1Z^{\otimes 16}, 32X^{\otimes 2}$ logicals= $\emptyset$

$QRM_4(4,4)$ $[[16,0,1]]$ S : $16X^{\otimes 1}$ logicals= $\emptyset$

$QRM_4(-1,0)$ $[[16,1,1]]$ S : $32Z^{\otimes 2}$ logicals= $\{Z, S, T, \ldots\}$

$QRM_4(0,1)$ $[[16,4,2]]$ S : $24Z^{\otimes 4}, 1X^{\otimes 16}$ logicals= $\{Z^{\otimes 2}, S^{\otimes 4}, T^{\otimes 8}, \sqrt{T}^{\otimes 16}\}$

$QRM_4(1,2)$ $[[16,6,4]]$ S : $8Z^{\otimes 8}, 8X^{\otimes 8}$ logicals= $\{H, Z^{\otimes 4}, S^{\otimes 16}\}$

$QRM_4(2,3)$ $[[16,4,2]]$ S : $1Z^{\otimes 16}, 24X^{\otimes 4}$ logicals= $\{Z^{\otimes 8}\}$

$QRM_4(3,4)$ $[[16,1,1]]$ S : $32X^{\otimes 2}$ logicals= $\{Z^{\otimes 16}\}$

$QRM_4(-1,1)$ $[[16,5,1]]$ logicals= $\{Z^{\otimes 1,2}, S^{\otimes 2,4}, T^{\otimes 4,8}, \sqrt{T}^{\otimes 8,16}, \sqrt[4]{T}^{\otimes 16}\}$

$QRM_4(0,2)$ $[[16,10,2]]$ logicals= $\{Z^{\otimes 2}, Z^{\otimes 4}, S^{\otimes 8}, S^{\otimes 16}\}$

$QRM_4(1,3)$ $[[16,10,2]]$ logicals= $\{Z^{\otimes 4}, Z^{\otimes 8}\}$

$QRM_4(2,4)$ $[[16,5,1]]$ logicals= $\{Z^{\otimes 8}, Z^{\otimes 16}\}$

$QRM_4(-1,2)$ $[[16,11,1]]$ logicals= $\{Z^{\otimes 1,2,4}, S^{\otimes 4,8,16}, T^{\otimes 16}\}$

$QRM_4(0,3)$ $[[16,14,2]]$ logicals= $\{H, Z^{\otimes 2,4,8}, S^{\otimes 16}\}$

$QRM_4(1,4)$ $[[16,11,1]]$ logicals= $\{Z^{\otimes 4}, Z^{\otimes 8}, Z^{\otimes 16}\}$

$QRM_4(-1,3)$ $[[16,15,1]]$ logicals= $\{Z^{\otimes 1,2,4,8}, S^{\otimes 8,16}\}$

$QRM_4(0,4)$ $[[16,15,1]]$ logicals= $\{Z^{\otimes 2}, Z^{\otimes 4}, Z^{\otimes 8}, Z^{\otimes 16}\}$

$QRM_4(-1,4)$ $[[16,16,1]]$ logicals= $\{H, Z, Z^{\otimes 2}, Z^{\otimes 4}, Z^{\otimes 8}, Z^{\otimes 16}, S^{\otimes 16}\}$

$QRM_{m=3}(q=1, r=1) \in [[n=8, k=0, d=4]]$

Our first encounter with a quantum error correcting code

The geometric construction of quantum Reed-Muller codes

The duality of non-Clifford operations

What about non-Reed-Muller codes?

- Modified Reed-Muller codes. $[[7,1,3]]$, a code with transversal Cliffords.
- Non-CSS codes. $[[5,1,3]]$, the smallest qubit QECC

A periodic table of classical trit generalized Reed-Muller codes

			1 $GRM_3(-1,0)$ $[1,0,\infty]_3$	$GRM_3(0,0)$ $[1,1,1]_3$			
		1 $GRM_3(-1,1)$ $[3,0,\infty]_3$	1 $GRM_3(0,1)$ $[3,1,3]_3$	1 $GRM_3(1,1)$ $[3,2,2]_3$	$GRM_3(2,1)$ $[3,3,1]_3$		
	1 $GRM_3(-1,2)$ $[9,0,\infty]_3$	2 $GRM_3(0,2)$ $[9,1,9]_3$	3 $GRM_3(1,2)$ $[9,3,6]_3$	2 $GRM_3(2,2)$ $[9,6,3]_3$	1 $GRM_3(3,2)$ $[9,8,2]_3$	$GRM_3(4,2)$ $[9,9,1]_3$	
1 $GRM_3(-1,3)$ $[27,0,\infty]_3$	3 $GRM_3(0,3)$ $[27,1,27]_3$	6 $GRM_3(1,3)$ $[27,4,18]_3$	7 $GRM_3(2,3)$ $[27,10,9]_3$	6 $GRM_3(3,3)$ $[27,17,6]_3$	3 $GRM_3(4,3)$ $[27,23,3]_3$	1 $GRM_3(5,3)$ $[27,26,2]_3$	$GRM_3(6,3)$ $[27,27,1]_3$
Trivial codes	Repetition codes					Single parity-check codes	Universe codes

One-qutrit quantum (stabilized) states

$GRM(-1,0)$ $[1,0,\infty]_3$	1	$GRM(0,0)$ $[1,1,1]_3$
$QGRM_0^3(-1,-1)$ $[[1,0,1]]_3$ $S = \{Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}\}$ codeword$= \|0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ logicals$= \emptyset$		$QGRM_0^3(0,0)$ $[[1,0,1]]_3$ $S = \{X = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\}$ codeword$= \|+\rangle = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ logicals$= \emptyset$
	$QGRM_0^3(-1,0)$ $[[1,1,1]]_3$ $S = \{\}$ codeword$=\|\psi\rangle$ logicals$=U$	

Three-qutrit quantum error (detection) codes

$GRM_3(-1,1)$ $[3,0,\infty]_3$	1	$GRM_3(0,1)$ $[3,1,3]_3$	1	$GRM_3(1,1)$ $[3,2,2]_3$	1	$GRM_3(2,1)$ $[3,3,1]_3$
$QGRM_1^3(-1,-1)$ $[[3,0,1]]_3$ $S =$ $\{Z_0, Z_1, Z_2\}$ codeword= $\lvert 0\rangle \lvert 0\rangle \lvert 0\rangle$ logicals= $\emptyset$		$QGRM_1^3(0,0)$ $[[3,0,2]]_3$ $S =$ $\{Z_0Z_1, Z_1Z_2, Z_2Z_0$ $X_0X_1X_2\}$ logicals= $\emptyset$		$QGRM_1^3(1,1)$ $[[3,0,2]]_3$ $S = \{Z_0Z_1Z_2,$ $X_0X_1, X_1X_2, X_2X_0\}$ logicals= $\emptyset$		$QGRM_1^3(2,2)$ $[[3,0,1]]_3$ $S =$ $\{X_0, X_1, X_2\}$ codeword= $\lvert +\rangle \lvert +\rangle \lvert +\rangle$ logicals= $\emptyset$
	$QGRM_1^3(-1,0)$ $[[3,1,1]]_3$ $S =$ $\{Z_0Z_1, Z_1Z_2, Z_2Z_0\}$ logicals= $\{XXX, Z, S, T, \dots\}$		$QGRM_1^3(0,1)$ $[[3,1,2]]_3$ $S = \{Z_0Z_1Z_2,$ $X_0X_1X_2\}$ logicals= $\{H, ZZ, SSS, XX, \sqrt{X}\sqrt{X}\sqrt{X}\}$		$QGRM_1^3(1,2)$ $[[3,1,1]]_3$ $S =$ $\{X_0X_1, X_1X_2, X_2X_0\}$ logicals= $\{ZZZ, X, \sqrt{X}, \sqrt[4]{X}, \dots\}$	
		$QGRM_1^3(-1,1)$ $[[3,2,1]]_3$ $S = \{Z_0Z_1Z_2\}$ logicals= $\{Z, ZZ, SS, SSS, XX, XXX\}$		$QGRM_1^3(0,2)$ $[[3,2,1]]_3$ $S = \{X_0X_1X_2\}$ logicals= $\{ZZ, ZZZ, X, XX, \sqrt{X}\sqrt{X}, \sqrt{X}\sqrt{X}\sqrt{X}\}$		
			$QGRM_1^3(-1,2)$ $[[3,3,1]]_3$ $S = \{\}$ logicals= $\{H, Z, ZZ, ZZZ, SSS, X, XX, XXX, \sqrt{X}\sqrt{X}\sqrt{X}\}$			

Nine-qutrit quantum error (correction) codes

$GRM_3(-1,2)$ $[9,0,\infty]_3$	1	$GRM_3(0,2)$ $[9,1,9]_3$	2	$GRM_3(1,2)$ $[9,3,6]_3$	3	$GRM_3(2,2)$ $[9,6,3]_3$	2	$GRM_3(3,2)$ $[9,8,2]_3$	1	$GRM_3(4,2)$ $[9,9,1]_3$
$QGRM_2^3(-1,-1)$ $[[9,0,1]]_3$ $S=$ $9Z^{\otimes 1}$ logicals= $\emptyset$		$QGRM_2^3(0,0)$ $[[9,0,2]]_3$ $S=$ $?Z^{\otimes 2}, 1X^{\otimes 9}$ logicals= $\emptyset$		$QGRM_2^3(1,1)$ $[[9,0,3]]_3$ $S=$ $6Z^{\otimes 3}, ?X^{\otimes 6}$ logicals= $\emptyset$		$QGRM_2^3(2,2)$ $[[9,0,3]]_3$ $S=$ $?Z^{\otimes 6}, 6X^{\otimes 3}$ logicals= $\emptyset$		$QGRM_2^3(3,3)$ $[[9,0,2]]_3$ $S=$ $1Z^{\otimes 9}, ?X^{\otimes 2}$ logicals= $\emptyset$		$QGRM_2^3(4,4)$ $[[9,0,1]]_3$ $S=$ $9X^{\otimes 1}$ logicals= $\emptyset$
	$QGRM_2^3(-1,0)$ $[[9,1,1]]_3$ $S=$ $?Z^{\otimes 2}$ logicals= $\{Z,S,T,\ldots\}$		$QGRM_2^3(0,1)$ $[[9,2,2]]_3$ $S=$ $6Z^{\otimes 3}, 1X^{\otimes 9}$ logicals= $\{Z^{\otimes 2}, S^{\otimes 3}, T^{\otimes 9}\}$		$QGRM_2^3(1,2)$ $[[9,3,3]]_3$ $S=$ $?Z^{\otimes 6}, ?X^{\otimes 6}$ logicals= $\{H, Z^{\otimes 3}, S^{\otimes 9}\}$		$QGRM_2^3(2,3)$ $[[9,2,2]]_3$ $S=$ $1Z^{\otimes 9}, 6X^{\otimes 3}$ logicals= $\{Z^{\otimes 6}\}$		$QGRM_2^3(3,4)$ $[[9,1,1]]_3$ $S=$ $?X^{\otimes 2}$ logicals= $\{Z^{\otimes 9}\}$	
		$QGRM_2^3(-1,1)$ $[[9,3,1]]_3$ logicals= $\{Z^{\otimes 1,2}, S^{\otimes 2,3}, T^{\otimes 6,9}\}$		$QGRM_2^3(0,2)$ $[[9,5,2]]_3$ logicals= $\{Z^{\otimes 2,3}, S^{\otimes 6,9}\}$		$QGRM_2^3(1,3)$ $[[9,5,2]]_3$ logicals= $\{Z^{\otimes 3,6}\}$		$QGRM_2^3(2,4)$ $[[9,3,1]]_3$ logicals= $\{Z^{\otimes 6,9}\}$		
			$QGRM_2^3(-1,2)$ $[[9,6,1]]_3$ logicals= $\{Z^{\otimes 1,2,3}, S^{\otimes 3,6,9}\}$		$QGRM_2^3(0,3)$ $[[9,7,2]]_3$ logicals= $\{H, Z^{\otimes 2,3,6}, S^{\otimes 9}\}$		$QGRM_2^3(1,4)$ $[[9,6,1]]_3$ logicals= $\{Z^{\otimes 3,6,9}\}$			
				$QGRM_2^3(-1,3)$ $[[9,8,1]]_3$ logicals= $\{Z^{\otimes 1,2,3,6}, S^{\otimes 6,9}\}$		$QGRM_2^3(0,4)$ $[[9,8,1]]_3$ logicals= $\{Z^{\otimes 2,3,6,9}\}$				
					$QGRM_2^3(-1,4)$ $[[9,9,1]]_3$ logicals= $\{H, Z^{\otimes 1,2,3,6,9}, S^{\otimes 9}\}$					

A periodic table of classical quart generalized Reed-Muller codes

$GRM_4(-1,0)$ $[1,0,\infty]_4$	1	$GRM_4(0,0)$ $[1,1,1]_4$

$GRM_4(-1,1)$ $[4,0,\infty]_4$	1	$GRM_4(0,1)$ $[4,1,4]_4$	1	$GRM_4(1,1)$ $[4,2,3]_4$	1	$GRM_4(2,1)$ $[4,3,2]_4$	1	$GRM_4(3,1)$ $[4,4,1]_4$

$GRM_4(-1,2)$ $[16,0,\infty]_4$	1	$GRM_4(0,2)$ $[16,1,16]_4$	2	$GRM_4(1,2)$ $[16,3,12]_4$	3	$GRM_4(2,2)$ $[16,6,8]_4$	4	$GRM_4(3,2)$ $[16,10,4]_4$	3	$GRM_4(4,2)$ $[16,13,3]_4$	2	$GRM_4(5,2)$ $[16,15,2]_4$	1	$GRM_4(6,2)$ $[16,16,1]_4$

Trivial codes

Repetition codes

Self-dual code

Single parity-check codes

Universe codes